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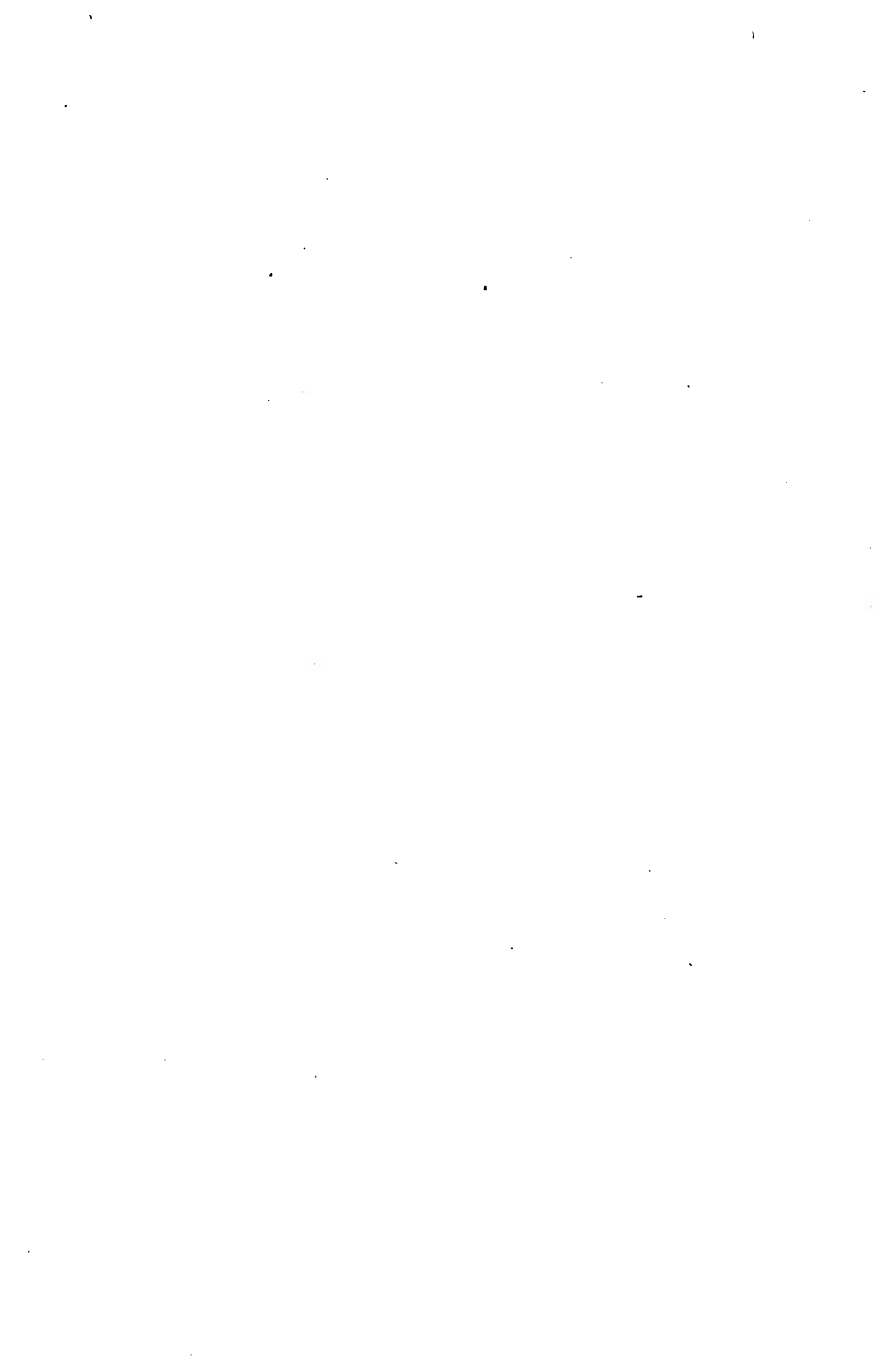
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°ELEMENTS OF ALGEBRA

BY

G. A. WENTWORTH, A.M.

AUTHOR OF A SERIES OF TEXT-BOOKS IN MATHEMATICS

COMPLETE EDITION

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PREFACE.

THE single aim in writing this volume has been to make an Algebra which the beginner would read with increasing interest, intelligence, and power. The fact has been kept constantly in mind that, to accomplish this object, the several parts must be presented so distinctly that the pupil will be led to feel that he is *mastering* the subject. Originality in a text-book of this kind is not to be expected or desired, and any claim to usefulness must be based upon the method of treatment and upon the number and character of the examples. About four thousand examples have been selected, arranged, and tested in the recitation-room, and any found too difficult have been excluded from the book. The idea has been to furnish a great number of examples for practice, but to exclude complicated problems that consume time and energy to little or no purpose.

In expressing the definitions, particular regard has been paid to brevity and perspicuity. The rules have been deduced from processes immediately preceding, and have been written, not to be committed to memory, but to furnish aids to the student in framing for himself intelligent statements of his methods. Each principle has been fully illustrated, and a sufficient number of problems has been given to fix it firmly in the pupil's mind before he proceeds to another. Many examples have been worked out, in order to exhibit the best methods of dealing with different classes of problems and the best arrangement of the work; and such aid has been given in the statement of problems as experience has shown to be necessary for the attainment of the best results. General demonstrations have been avoided whenever a particular illustration would serve the purpose, and the application of the principle to similar cases was obvious. The reason for this course is, that

the pupil must become familiar with the separate steps from particular examples, before he is able to follow them in a general demonstration, and to understand their logical connection.

It is presumed that pupils will have a fair acquaintance with Arithmetic before beginning the study of Algebra; and that sufficient time will be afforded to learn the *language* of Algebra, and to settle the principles on which the ordinary processes of Algebra are conducted, before attacking the harder parts of the book. "Make haste slowly" should be the watchword for the early chapters.

It has been found by actual trial that a class can accomplish the whole work of this Algebra in a school year, with one recitation a day; and that the student will not find it so difficult as to discourage him, nor yet so easy as to deprive him of the rewards of patient and successful labor. At least one-fourth of the year is required to reach the chapter on Fractions; but, if the first hundred pages are thoroughly mastered, rapid and satisfactory progress will be made in the rest of the book.

Particular attention should be paid to the chapter on Factoring; for a thorough knowledge of this subject is requisite to success in common algebraic work.

Attention is called to the method of presenting Choice and Chance. The accomplished mathematician may miss the elegance of the *general* method usually adopted in Algebras; but it is believed that this mode of treatment will furnish to the average student the only way by which he can arrive at an understanding of the principles underlying these difficult subjects. In the preparation of these chapters the author has had the assistance and coöperation of G. A. Hill, A.M., of Cambridge, Mass., to whom he gratefully acknowledges his obligation.

The chapter on the General Theory of Equations has been contributed by Professor H. A. Howe of Denver University, Colorado.

The materials for this Algebra have been obtained from English, German, and French sources. To avoid trespassing upon the works of recent American authors, no American text-book has been consulted.

The author returns his sincere thanks for assistance to Rev. Dr. Thomas Hill; to Professors Samuel Hart of Hartford, Ct.; C. H. Judson of Greenville, S.C.; O. S. Westcott of Racine, Wis.; G. B. Halsted of Princeton, N.J.; M. W. Humphreys of Nashville, Tenn.; W. LeConte Stevens of New York, N.Y.; G. W. Bailey of New York, N.Y.; Robert A. Benton of Concord, N.H.; and to Dr. D. F. Wells of Exeter.

There will be two editions of the Algebra: one of 350 pages, designed for high schools and academies, will contain an ample amount to meet the requirements for admission to any college; the other will consist of the Elementary part and about 150 pages more, and will include the subjects usually taught in colleges.

Answers to the problems are bound separately in paper covers, and will be furnished free to pupils when *teachers* apply to the *publishers* for them.

Any corrections or suggestions relating to the work will be thankfully received.

G. A. WENTWORTH.

EXETER, N.H.,
May, 1881.

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ELEMENTS OF ALGEBRA.



CHAPTER I.

QUANTITY AND NUMBER.

1. **WHATEVER** may be regarded as being made up of parts like the whole is called a **Quantity**.

2. To **measure** a quantity of any kind is to find how many times it contains another *known quantity of the same kind*.

3. A *known quantity* which is adopted as a standard for measuring quantities of the same kind is called a **Unit**. Thus, the foot, the pound, the dollar, the day, are units for measuring distance, weight, money, time.

4. A **Number** arises from the repetitions of the unit of measure, and shows *how many times* the unit is contained in the quantity measured.

5. When a quantity is measured, the result obtained is expressed by prefixing to the *name* of the unit the *number* which shows how many times the unit is contained in the quantity measured; and the two combined denote a quantity expressed in units. Thus, 7 feet, 8 pounds, 9 dollars, 10 days, are quantities expressed in their respective units.

When a question about a quantity includes the unit, the answer is a *number*; when it does not include the unit, the answer is a *quantity*. Thus, if a man who has fifteen bushels of wheat be asked *how many bushels* of wheat he has, the answer is the *number*, fifteen; if he be asked *how much* wheat he has, the answer is the *quantity*, fifteen bushels.

A number answers the question, How many? a quantity, the question, How much?

NUMBERS.

6. The symbols which **Arithmetic** employs to represent numbers are the figures 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. The *natural* series of numbers begins with 0; each succeeding number is obtained by adding one to the preceding number, and the series is infinite.

7. Besides figures, the chief symbols used in Arithmetic are :

- + (read, plus), the sign of addition.
- (read, minus), the sign of subtraction.
- × (read, multiplied by), the sign of multiplication.
- ÷ (read, divided by), the sign of division.
- = (read, is equal to), the sign of equality.

EXERCISE. — Read :

$7 + 12 = 19.$	$8 + 3 - 5 = 20 - 15 + 1.$
$9 - 4 = 5.$	$24 + 6 = 10 \times 3.$
$6 \times 4 = 24.$	$14 - 7 + 5 = 6 \times 2.$
$48 \div 3 = 16.$	$9 \times 5 = 180 \div 4.$

8. Any figure, or combination of figures, as 7, 28, 346, has one, and only one, value. That is, figures represent

particular numbers. But numbers possess many *general properties*, which are true, not only of a particular number, but of all numbers.

Thus, the sum of 12 and 8 is 20, and the difference between 12 and 8 is 4. Their sum added to their difference is 24, which is twice the greater number. Their difference taken from their sum is 16, which is twice the smaller number.

9. As this is true of any two numbers, we have this general property: *The sum of two numbers added to their difference is twice the greater number; the difference of two numbers taken from their sum is twice the smaller number.* Or,

1. (greater number + smaller number) + (greater number - smaller number) = twice greater number.
2. (greater number + smaller number) - (greater number - smaller number) = twice smaller number.

But these statements may be very much shortened; for, as greater number and smaller number may mean any two numbers, two letters, as a and b , may be used to represent them; and $2a$ may represent twice the greater, and $2b$ twice the smaller. Then these statements become:

1. $(a + b) + (a - b) = 2a$.
2. $(a + b) - (a - b) = 2b$.

In studying the general properties of numbers, letters may represent any numerical values consistent with the conditions of the problem.

10. It is also convenient to use letters to denote numbers which are *unknown*, and which are to be found from certain given relations to other known numbers.

Thus, the solution of the problem, "Find two numbers such that, when the greater is divided by the less, the quotient is 4, and the remainder 3; and when the sum of the two numbers is increased by 38, and the result divided by the greater of the two numbers, the quotient is 2 and the remainder 2," is much simplified by the use of letters to represent the unknown numbers.

11. The science which employs letters in reasoning about numbers, either to discover their *general properties*, or to find the value of an *unknown number* from its relations to known numbers, is called **Algebra**.

ALGEBRAIC NUMBERS.

12. There are quantities which stand to each other in such opposite relations that, when we combine them, they cancel each other entirely or in part. Thus, six dollars *gain* and six dollars *loss* just cancel each other; but ten dollars *gain* and six dollars *loss* cancel each other only in part. For the six dollars *loss* will cancel six dollars of the *gain* and will leave four dollars.

An opposition of this kind exists in *assets* and *debts*, in *income* and *outlay*, in motion *forwards* and *backwards*, in motion *to the right* and *to the left*, in time *before* and *after* a fixed date, in the degrees *above* and *below* zero on a thermometer.

From this relation of quantities a question often arises which is not considered in Arithmetic; namely, the subtracting of a greater number from a smaller. This cannot be done in Arithmetic, for the real nature of subtraction consists in *counting backwards*, along the natural series of numbers. If we wish to subtract four from six, we start at six in the natural series, count four units backwards, and

arrive at two, the difference sought. If we subtract six from six, we start at six in the natural series, count six units backwards, and arrive at zero. If we try to subtract nine from six, we cannot do it, because, when we have counted backwards as far as zero, *the natural series of numbers comes to an end.*

13. In order to subtract a greater number from a smaller it is necessary to *assume* a new series of numbers, beginning at zero and extending to the left of zero. The series to the left of zero must ascend from zero by the repetition of the unit, precisely like the natural series to the right of zero; and the *opposition* between the right-hand series and the left-hand series must be clearly marked. This opposition is indicated by calling every number in the right-hand series a *positive* number, and prefixing to it, when written, the sign $+$; and by calling every number in the left-hand series a *negative* number, and prefixing to it the sign $-$. The two series of numbers will be written thus :

$$\cdots -4, -3, -2, -1, \quad 0, +1, +2, +3, +4, \cdots$$

If, now, we wish to subtract 9 from 6, we begin at 6 in the positive series, count nine units in the *negative direction* (to the left), and arrive at -3 in the negative series. That is $6 - 9 = -3$.

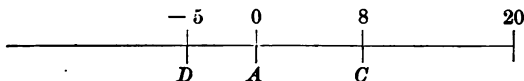
The result obtained by subtracting a greater number from a less, when both are positive, is *always a negative number.*

If a and b represent any two numbers of the positive series, the expression $a - b$ will denote a positive number when a is greater than b ; will be equal to zero when a is equal to b ; will denote a negative number when a is less than b .

If we wish to add 9 to -6 , we begin at -6 , in the

negative series, count nine units in the *positive direction* (to the right), and arrive at $+3$, in the positive series.

We may illustrate the use of positive and negative numbers as follows :



Suppose a person starting at A walks 20 feet to the right of A , and then returns 12 feet, where will he be? Answer: at C , a point 8 feet to the right of A . That is, 20 feet $-$ 12 feet $=$ 8 feet; or $20 - 12 = 8$.

Again, suppose he walks from A to the right 20 feet, and then returns 20 feet, where will he be? Answer: at A , the point from which he started. That is, $20 - 20 = 0$.

Again, suppose he walks from A to the right 20 feet, and then returns 25 feet, where will he now be? Answer: at D , a point 5 feet to the left of A . That is, if we consider distance measured in feet to the left of A as forming a negative series of numbers, beginning at A , $20 - 25 = -5$. Hence, the phrase, 5 feet to the left of A , is now expressed by the negative number -5 .

14. Numbers provided with the sign $+$ or $-$ are called **algebraic numbers**. They are unknown in Arithmetic, but play a very important part in Algebra. In contradistinction, numbers not affected by the signs $+$ or $-$ are termed **absolute numbers**.

15. Every algebraic number, as $+4$ or -4 , consists of a sign $+$ or $-$ and the absolute value of the number; in this case 4. The sign shows whether the number belongs to the positive or negative series of numbers; the absolute value shows what place the number has in the positive or negative series.

16. When no sign stands before a number, the sign $+$ is always understood; thus, 4 means the same as $+4$, a means the same as $+a$. But *the sign $-$ is never omitted.*

17. Two numbers which have one the sign $+$ and the other the sign $-$, are said to have **unlike signs**.

18. Two numbers which have the same absolute values, but unlike signs, always cancel each other when combined; thus $+4 - 4 = 0$, $+a - a = 0$.

19. The use of the signs $+$ and $-$, to indicate addition and subtraction, must be carefully distinguished from their use to indicate in which series, the positive or the negative, a given number belongs. In the first sense, they are signs of *operations*, and are common both to Arithmetic and Algebra. In the second sense, they are signs of *opposition*, and are employed in Algebra alone.

FACTORS AND POWERS.

20. When a number consists of the product of two or more numbers, each of these numbers is called a **factor** of the product.

When these numbers are denoted by letters, the sign \times is omitted; thus, instead of $a \times b$, we write ab ; instead of $a \times b \times c$, we write abc .

The expression abc must not be confounded with $a+b+c$; the first is a product, the second is a sum. If $a=2$, $b=3$, $c=4$, then

$$\begin{aligned} abc &= 2 \times 3 \times 4 = 24; \\ a + b + c &= 2 + 3 + 4 = 9. \end{aligned}$$

21. Factors expressed by letters are called **literal factors**; factors expressed by figures are called **numerical factors**.

22. A known factor of a product which is prefixed to another factor, to show how many times that factor is taken, is called a **coefficient**. Thus, in $7c$, 7 is the coefficient of c ; in $7ax$, 7 is the coefficient of ax , or, if a be known, $7a$ is the coefficient of x . When no numerical coefficient occurs in a product, 1 is always understood. Thus, ax means the same as $1ax$.

23. A product consisting of two or more **equal** factors is called a **power** of that factor.

24. The **index** or **exponent** of a **power** is a small figure or letter placed at the right of a number, to show how many times the number is taken as a factor. Thus, a^1 , or simply a , denotes that a is taken once as a factor; a^2 denotes that a is taken twice as a factor; a^3 denotes that a is taken three times as a factor; and a^n denotes that a is taken n times as a factor. These are read: the first power of a ; the second power of a ; the third power of a ; the n th power of a .

a^3 is written instead of aaa .

a^n is written instead of aaa , etc., repeated n times.

The meaning of coefficient and exponent must be carefully distinguished. Thus,

$$4a = a + a + a + a;$$

$$a^4 = a \times a \times a \times a.$$

$$\text{If } a = 3, \quad 4a = 3 + 3 + 3 + 3 = 12.$$

$$a^4 = 3 \times 3 \times 3 \times 3 = 81.$$

25. The second power of a number is generally called the *square* of that number; thus, a^2 is called the *square* of a , because if a denote the number of units of length in the side of a square, a^2 denotes the number of units of surface in the square.

The third power of a number is generally called the *cube* of that number; thus, a^3 is called the *cube* of a , because if a denote the number of units of length in the edge of a cube, a^3 denotes the number of units of volume in the cube.

ALGEBRAIC SYMBOLS.

26. Known numbers in Algebra are denoted by figures and by the first letters of some alphabet; as a, b, c , etc.; a', b', c' , read *a prime, b prime, c prime*, etc.; a_1, b_1, c_1 , read *a one, b one, c one*.

Unknown numbers are generally known by the last letters of some alphabet; as x, y, z, x', y', z' , etc.

27. The **symbols of operations** are the same in Algebra as in Arithmetic. One point of difference, however, must be carefully observed. When a symbol of operation is omitted in the notation of Arithmetic, it is always the *symbol of addition*: but when a symbol of operation is omitted in the notation of Algebra, it is always the *symbol of multiplication*. Thus, 456 means $400 + 50 + 6$, but $4ab$ means $4 \times a \times b$; $4\frac{1}{2}$ means $4 + \frac{1}{2}$, but $4\frac{a}{b}$ means $4 \times \frac{a}{b}$.

28. The **symbols of relation** are $=, >, <$, which stand for the words "is equal to," "is greater than," and "is less than," respectively.

29. The **symbols of aggregation** are the bar, $|$; the vinculum, $—$; the parenthesis, $()$; the bracket, $[]$; and the brace, $\{ \}$. Thus, each of the expressions, $\overbrace{x+y}^x$, $\overline{x+y}$, $(x+y)$, $[x+y]$, $\{x+y\}$, signifies that $x+y$ is to be treated as a single number.

30. The **symbols of continuation** are dots,, or dashes, -----, and are read, "and so on."

31. The **symbol of deduction** is \therefore , and is read "hence," or "therefore."

ALGEBRAIC EXPRESSIONS.

32. An **algebraic expression** is any number written in algebraic symbols. Thus, $8c$ is the algebraic expression for 8 times the number denoted by c .

If an algebraic expression contains only *integral forms*, that is, contains no *letter in the denominator* of any of its terms, it is called an **integral expression**.

33. A **term** is an algebraic expression the parts of which are not separated by the sign of addition or subtraction. Thus, $3ab$, $5xy$, $3ab \div 5xy$ are terms.

34. A **monomial** or **simple expression** is an expression which contains only one term.

35. A **polynomial** or **compound expression** is an expression which contains two or more terms. A **binomial** is a polynomial of two terms. A **trinomial** is a polynomial of three terms.

36. **Like terms** are terms which have the same letters, and the corresponding letters affected by the same exponents. Thus, $7a^2cx^3$ and $-5a^2cx^3$ are like terms; but $7a^2cx^3$ and $-ac^2x^3$ are unlike terms.

37. The **dimensions** of a term are its literal factors.

38. The **degree** of a term is equal to the number of its dimensions, and is found by taking the sum of the exponents of its literal factors. Thus, $3xy$ is of the *second* degree, and $5x^2yz^3$ is of the *sixth* degree.

39. A polynomial is said to be **homogeneous** when all its terms are of the same degree. Thus, $7x^3 - 5x^2y + xyz$ is homogeneous, for each term is of the third degree.

40. A polynomial is said to be **arranged** according to the powers of some letter when the exponents of that letter either descend or ascend in order of magnitude. Thus, $3ax^3 - 4bx^2 - 6ax + 8b$ is arranged according to the descending powers of x , and $8b - 6ax - 4bx^2 + 3ax^3$ is arranged according to the ascending powers of x .

41. The **numerical value** of an algebraic expression is the number obtained by giving a particular value to each letter, and then performing the operations indicated.

42. Two numbers are **reciprocals** of each other when their product is equal to unity. Thus, a and $\frac{1}{a}$ are reciprocals.

AXIOMS.

43. 1. Things which are equal to the same thing are equal to each other.

2. If equal numbers be added to equal numbers, the sums will be equal.

3. If equal numbers be subtracted from equal numbers, the remainders will be equal.

4. If equal numbers be multiplied into equal numbers, the products will be equal.

5. If equal numbers be divided by equal numbers, the quotients will be equal.

6. If the same number be both added to and subtracted from another, the value of the latter will not be altered.

7. If a number be both multiplied and divided by another, the value of the former will not be altered.

EXERCISE 1.

If $a=1$, $b=2$, $c=3$, $d=4$, $e=5$, $f=0$, find the numerical values of the following expressions :

$$1. 9a + 2b + 3c - 2f. \quad 4. \frac{4ac}{b} + \frac{8bc}{d} - \frac{5cd}{e}.$$

$$2. 4e - 3a - 3b + 5c. \quad 5. 7e + bcd - \frac{3bde}{2ac}.$$

$$3. 8abc - bcd + 9cde - def. \quad 6. abc^2 + bcd^2 - dea^2 + f^3.$$

$$7. e^4 + 6e^2b^2 + b^4 - 4e^2b - 4eb^3.$$

$$8. \frac{8a^2 + 3b^2}{a^2b^2} + \frac{4c^2 + 6b^2}{c^2 - b^2} - \frac{c^2 + d^2}{e^2}.$$

$$9. \frac{d^e}{b^e}. \quad 11. \frac{b^e + d^e}{b^2 + d^2 - bd}.$$

$$10. \frac{e^e + b^a}{e^b - b^e}. \quad 12. \frac{e^e - d^e}{e^2 + ed + d^2}.$$

In simplifying compound expressions, each term must be reduced to its simplest form before the operations of addition and subtraction are performed.

In simplifying a term, omit the sign \times when possible, and write indicated divisions in the form of fractions.

Simplify the following expressions :

$$13. 100 + 80 \div 4. \quad 15. 25 + 5 \times 4 - 10 \div 5.$$

$$14. 75 - 25 \times 2. \quad 16. 24 - 5 \times 4 \div 10 + 3.$$

$$17. (24 - 5) \times (4 \div 10 + 3).$$

Find the numerical value of the following expressions, in which $a=2$, $b=10$, $x=3$, $y=5$:

$$18. xy + 4a \times 2. \quad 20. 3x + 7y \div 7 + a \times y.$$

$$19. xy - 15b \div 5. \quad 21. 6b - 8y \div 2y \times b - 2b.$$

$$22. (6b - 8y) \div 2y \times b + 2b.$$

$$23. (6b - 8y) \div (2y \times b) + 2b.$$

$$24. 6b - (8y \div 2y) \times b - 2b.$$

$$25. 6b \div (b - y) - 3x + bxy \div 10a.$$

ALGEBRAIC NOTATION.

26. Express the sum of a and b .
27. Express the double of x .
28. By how much is a greater than 5?
29. If x is a whole number, what is the next number above it?
30. Write five numbers in order of magnitude, so that x shall be the middle number.
31. What is the sum of $x + x + x + \dots$ written a times?
32. If the product is xy and the multiplier x , what is the multiplicand?
33. A man who has a dollars spends b dollars; how many dollars has he left?
34. A regiment of men can be drawn up in a ranks of b men each, and there are c men over; of how many men does the regiment consist?
35. Write, the sum of x and y divided by c is equal to the product of a , b , and m , diminished by six times c , and increased by the quotient of a divided by the sum of x and y .
36. Write, six times the square of n , divided by m minus a , increased by five b into the expression c plus d minus a .
37. Write, four times the fourth power of a , diminished by five times the square of a into the square of b , and increased by three times the fourth power of b .

EXERCISE 2.

That the beginner may see how Algebra is employed in the solution of problems, the following simple exercises are introduced :

1. John and James together have \$6. James has twice as much as John. How much has each ?

Let x denote the *number* of dollars John has.

Then $2x =$ the *number* of dollars James has,

and $x + 2x =$ the number of dollars both have.

But $6 =$ the number of dollars both have ;

$$\therefore x + 2x = 6,$$

$$\text{or} \qquad 3x = 6,$$

$$\text{and} \qquad x = 2.$$

Therefore, John has \$2, and James has \$4.

2. A stick of timber 40 feet long is sawed in two, so that one part is two-thirds as long as the other. Required the length of each part.

Let $3x$ denote the *number* of feet in the longer part.

Then $2x =$ the *number* of feet in the shorter part,

and $3x + 2x =$ the number of feet in both together.

But $40 =$ the number of feet in both together ;

$$\therefore 3x + 2x = 40,$$

$$\text{or} \qquad 5x = 40,$$

$$\text{and} \qquad x = 8.$$

Therefore, the longer part, or $3x$, is 24 feet long ; and the shorter, or $2x$, is 16 feet.

NOTE. The *unit* of the quantity sought is always given, and only the **number** of such units is required. Therefore x must never be put for *money, length, time, weight, etc.*, but always for the required **number of specified units** of money, length, time, weight, etc.

The beginner should give particular attention to this caution.

3. The greater of two numbers is six times the smaller, and their sum is 35. Required the numbers.
4. Thomas had 75 cents. After spending a part of his money, he found he had twice as much left as he had spent. How much had he spent?
5. A tree 75 feet high was broken, so that the part broken off was four times the length of the part left standing. Required the length of each part.
6. Four times the smaller of two numbers is three times the greater, and their sum is 63. Required the numbers.
7. A farmer sold a sheep, a cow, and a horse, for \$216. He sold the cow for seven times as much as the sheep, and the horse for four times as much as the cow. How much did he get for each?
8. George bought some apples, pears, and oranges, for 91 cents. He paid twice as much for the pears as for the apples, and twice as much for the oranges as for the pears. How much money did he spend for each?
9. A man bought a horse, wagon, and harness, for \$350. He paid for the horse four times as much as for the harness, and for the wagon one-half as much as for the horse. What did he pay for each?
10. Distribute \$3 among Thomas, Richard, and Henry, so that Thomas and Richard shall each have twice as much as Henry.
11. Three men, A, B, and C, pay \$1000 taxes. B pays 4 times as much as A, and C an amount equal to the sum of what the other two pay. How much does each pay?

CHAPTER II.

ADDITION AND SUBTRACTION.

44. An algebraic number which is to be added or subtracted is often inclosed in a parenthesis, in order that the signs $+$ and $-$ which are used to distinguish positive and negative numbers may not be confounded with the $+$ and $-$ signs that denote the operations of addition and subtraction. Thus, $+4 + (-3)$ expresses the sum, and $+4 - (-3)$ expresses the difference, of the numbers $+4$ and -3 .

45. In order to add two algebraic numbers, we begin at *the place in the series which the first number occupies*, and count, *in the direction indicated by the sign of the second number*, as many units as are equal to the absolute value of the second number. Thus, the sum of $+4 + (+3)$ is found by counting from $+4$ three units in the *positive* direction, and is, therefore, $+7$; the sum of $+4 + (-3)$ is found by counting from $+4$ three units in the *negative* direction, and is, therefore, $+1$.

In like manner, the sum of $-4 + (+3)$ is -1 , and the sum of $-4 + (-3)$ is -7 . That is,

$$(1) +4 + (+3) = 7; \quad (3) -4 + (+3) = -1;$$

$$(2) +4 + (-3) = 1; \quad (4) -4 + (-3) = -7.$$

I. Therefore, to add two numbers with **like** signs, *find the sum of their absolute values, and prefix the common sign to the sum.*

II. To add two numbers with **unlike** signs, *find the difference of their absolute values, and prefix the sign of the greater number to the difference.*

EXERCISE 3.

1. $+16 + (-11) =$ 3. $+68 + (-79) =$
2. $-15 + (-25) =$ 4. $-7 + (+4) =$
5. $+33 + (+18) =$
6. $+378 + (+709) + (-592) =$
7. A man has \$5242 and owes \$2758. How much is he worth?
8. The First Punic War began B.C. 264, and lasted 23 years. When did it end?
9. Augustus Cæsar was born B.C. 63, and lived 77 years. When did he die?
10. A man goes 65 steps forwards, then 37 steps backwards, then again 48 steps forwards. How many steps did he take in all? How many steps is he from where he started?

ADDITION OF MONOMIALS.

46. If a and b denote the absolute values of any two numbers, 1, 2, 3, 4 (§ 45) become :

- (1) $+a + (+b) = a + b$; (3) $-a + (+b) = -a + b$;
- (2) $+a + (-b) = a - b$; (4) $-a + (-b) = -a - b$.

Therefore, to add two terms, *write them one after the other with unchanged signs.*

It should be noticed that the order of the terms is immaterial. Thus, $+a - b = -b + a$. If $a = 8$ and $b = 12$, the result in either case is -4 .

$$47. \quad \begin{aligned} 3a + 5a + 2a + 6a + a &= 17a. \\ -2c - 3c - c - 4c - 8c &= -18c. \end{aligned}$$

Therefore, to add several **like terms** which have the **same**

sign, add the coefficients, prefix the common sign, and annex the common symbols.

$$\begin{aligned} 48. \quad 7a - 6a + 11a + a - 5a - 2a &= 19a - 13a = 6a. \\ -3a - 15a - 7a + 14a - 2a &= 14a - 27a = -13a. \end{aligned}$$

Therefore, to add several like terms which have **not all the same sign**, find the difference between the sum of the positive coefficients and the sum of the negative coefficients, prefix the sign of the greater sum, and annex the common symbols.

$$\begin{aligned} 49. \quad 5a - 2b + 3a &= 8a - 2b. \\ -3ax + 8y + 9ax - 4c &= 6ax + 8y - 4c. \end{aligned}$$

Therefore, to add terms which are **not all like terms**, combine the like terms, and write down the other terms, each preceded by its proper sign.

EXERCISE 4.

1. $5ab + (-5ab) =$
2. $8mx + (-2mx) =$
3. $-13mng + (-7mng) =$
4. $-5x^2 + (+8x^2) =$
5. $25my^2 + (-18my^2) =$
6. $7ab + (-5ab) =$
7. $120my + (-95my) =$
8. $-33ab^2 + (+11ab^2) =$
9. $-75xy + (+20xy) =$
10. $+15a^2x^2 + (-a^2x^2) =$
11. $-b^2m^3 + (+7b^2m^3) =$
12. $5a + (-3b) + (+4a) + (-7b) =$
13. $4a^2c + (-10xyz) + (+6a^2c) + (-9xyz)$
 $+ (-11a^2c) + (+20xyz) =$
14. $3x^2y + (-4ab) + (-2mn) + (+5x^2y)$
 $+ (-x^2y) + (-4x^2y) =$

ADDITION OF POLYNOMIALS.

50. Two or more polynomials are added by adding their separate terms.

It is convenient to arrange the terms in columns, so that like terms shall stand in the same column. Thus,

$$\begin{array}{rcl}
 (1) & 2a^3 - 3a^2b + 4ab^2 + b^3 & (2) - 2x^2y + 6y^2 - 1 \\
 & a^3 + 4a^2b - 7ab^2 - 2b^3 & - 4x^2y + 2xy^2 + 5 \\
 & - 3a^3 + a^2b - 3ab^2 - 4b^3 & 6x^2y + 2 \\
 & 2a^3 + 2a^2b + 6ab^2 - 3b^3 & x^2y - y^3 \\
 \hline
 & 2a^3 + 4a^2b - 8b^3 & - 2x^2y - 5 \\
 & & - x^2y + 2xy^2 + 5y^2 + 1
 \end{array}$$

EXERCISE 5.

Add:

- $5a + 3b + c, 3a + 3b + 3c, a + 3b + 5c.$
- $7a - 4b + c, 6a + 3b - 5c, -12a + 4c.$
- $a + b - c, b + c - a, c + a - b, a + b - c.$
- $a + 2b + 3c, 2a - b - 2c, b - a - c, c - a - b.$
- $a - 2b + 3c - 4d, 3b - 4c + 5d - 2a,$
 $5c - 6d + 3a - 4b, 7d - 4a + 5b - 4c.$
- $x^3 - 4x^2 + 5x - 3, 2x^3 - 7x^2 - 7x^2 - 14x + 5,$
 $-x^3 + 9x^2 + x + 8.$
- $x^4 - 2x^3 + 3x^2, x^3 + x^2 + x, 4x^4 + 5x^3,$
 $2x^2 + 3x - 4, -3x^2 - 2x - 5.$
- $a^3 + 3ab^2 - 3a^2b - b^3, 2a^3 + 5a^2b - 6ab^2 - 7b^2,$
 $a^3 - ab^2 + 2b^3.$
- $2ab - 3ax^2 + 2a^2x, 12ab - 6a^2x + 10ax^2,$
 $ax^3 - 8ab - 5a^2x.$

10. $c^4 - 3c^3 + 2c^2 - 4c + 7$, $2c^4 + 3c^3 + 2c^2 + 5c + 6$,
 $-4c^4 - 4c^3 - 5$.
11. $3x^2 - xy + xz - 3y^2 + 4yz - z^2$, $-5x^2 - xy - xz + 5yz$
 $6x^2 - 6y - 6z$, $4yz - 5yz + 3z^2$,
 $-4x^2 + y^2 + 3yz + 3z^2$.
12. $m^5 - 3m^4n - 6m^3n^2$, $+m^3n^3 + m^2n^3 - 5m^4n$,
 $7m^3n^2 + 4m^2n^3 - 3mn^4$, $-2m^2n^3 - 3mn^4 + 4n^5$,
 $2mn^4 + 2n^5 + 3m^5$, $-n^5 + 2m^5 + 7m^4n$.

SUBTRACTION.

51. In order to find the difference between two algebraic numbers, we begin *at the place in the series which the minuend occupies*, and count in the direction opposite to that indicated by the sign of the subtrahend as many units as are equal to the absolute value of the subtrahend.

Thus, the difference between $+4$ and $+3$ is found by counting from $+4$ three units in the *negative* direction, and is, therefore, $+1$; the difference between $+4$ and -3 is found by counting from $+4$ three units in the *positive* direction, and is, therefore, $+7$.

In like manner, the difference between -4 and $+3$ is -7 ; the difference between -4 and -3 is -1 .

Compare these results with results obtained in addition:

(1) $+4 - (+3) = 1$	$+4 + (-3) = 1$.
(2) $+4 - (-3) = 7$	$+4 + (+3) = 7$.
(3) $-4 - (+3) = -7$	$-4 + (-3) = -7$.
(4) $-4 - (-3) = -1$	$-4 + (+3) = -1$.

Or, (1) $+4 - (+3) = +4 + (-3)$.

(2) $+4 - (-3) = +4 + (+3)$.

(3) $-4 - (+3) = -4 + (-3)$.

(4) $-4 - (-3) = -4 + (+3)$.

52. From (1) and (3), it is evident that *subtracting a positive number is equivalent to adding an equal negative number.*

From (2) and (4), it is evident that *subtracting a negative number is equivalent to adding an equal positive number.*

To subtract, therefore, one algebraic number from another, *change the sign of the subtrahend, and then add the subtrahend to the minuend.*

EXERCISE 6.

$$1. +25 - (+16) =$$

$$3. -31 - (+58) =$$

$$2. -50 - (-25) =$$

$$4. +107 - (-93) =$$

5. Rome was ruled by emperors from B.C. 30 to its fall, A.D. 476. How long did the empire last?

6. The continent of Europe lies between 36° and 71° north latitude, and between 12° west and 63° east longitude (from Paris). How many degrees does it extend in latitude, and how many in longitude?

SUBTRACTION OF MONOMIALS.

If a and b denote the absolute values of any two numbers, 1, 2, 3, and 4 (§ 51) become :

$$(1) +a - (+b) = a - b.$$

$$(3) -a - (+b) = -a - b.$$

$$(2) +a - (-b) = a + b.$$

$$(4) -a - (-b) = -a + b.$$

To subtract, therefore, one term from another, *change the sign of the term to be subtracted, and write the terms one after the other.*

EXERCISE 7.

1. $5x - (-4x) =$
2. $-3ab - (+5ab) =$
3. $3ab^2 - (+10ab^2) =$
4. $15m^2x^2 - (-7m^2x^2) =$
5. $-7ay - (-3ay) =$
6. $17ax^2 - (-24ax^2) =$
7. $5a^2x - (-3a^2x) =$
8. $-4xy - (-5xy) =$
9. $8ax - (-3ay) =$
10. $2ab^2y - (+aby) =$
11. $9x^2 + (5x^2) - (+8x^2) =$
12. $5x^2y - (-18x^2y) + (-10x^2y) =$
13. $17ax^2 - (-ax^2) - (+24ax^2) =$
14. $-3ab + (2mx) - (-4mx) =$
15. $3a - (+2b) - (-4c) =$

SUBTRACTION OF POLYNOMIALS.

53. When one polynomial is to be subtracted from another, place its terms under the like terms of the other, if they have like terms, change the signs of the subtrahend, and add.

$$\begin{array}{r} \text{From} \quad 4x^3 - 3x^2y - xy^2 + 2y^3 \\ \text{take} \quad 2x^3 - x^2y + 5xy^2 - 3y^3 \\ \hline \end{array}$$

Change the signs of the subtrahend and add :

$$\begin{array}{r} 4x^3 - 3x^2y - xy^2 + 2y^3 \\ - 2x^3 + x^2y - 5xy^2 - 3y^3 \\ \hline 2x^3 - 2x^2y - 6xy^2 + 5y^3 \end{array}$$

$$\begin{array}{r} \text{From} \quad a^3x^2 + 2a^2x^3 - 4ax^4 \\ \text{take} \quad a^5 + 4a^3x^2 - 3a^2x^3 - 4ax^4 \\ \hline -a^5 - 3a^3x^2 + 5a^2x^3 \end{array}$$

In the last example we have conceived the signs to be changed without actually changing them. The beginner should do the examples by both methods until he has acquired sufficient practice, when he should use the second method only.

EXERCISE 8.

1. From $6a - 2b - c$ take $2a - 2b - 3c$.
2. From $3a - 2b + 3c$ take $2a - 7b - c - b$.
3. From $7x^2 - 8x - 1$ take $5x^2 - 6x + 3$.
4. From $4x^4 - 3x^3 - 2x^2 - 7x + 9$
take $x^4 - 2x^3 - 2x^2 + 7x - 9$.
5. From $2x^2 - 2ax + 3a^2$ take $x^2 - ax + a^2$.
6. From $x^2 - 3xy - y^2 + yz - 2z^2$
take $x^2 + 2xy + 5xz - 3y^2 - 2z^2$.
7. From $a^3 - 3a^2b + 3ab^2 - b^3$
take $-a^3 + 3a^2b - 3ab^2 + b^3$.
8. From $x^2 - 5xy + xz - y^2 + 7yz + 2z^2$
take $x^2 - xy - xz + 2yz + 3z^2$.
9. From $2ax^2 + 3abx - 4b^2x + 12b^3$
take $ax^2 - 4abx + bx^2 - 5b^2x - x^3$.
10. From $6x^3 - 7x^2y + 4xy^2 - 2y^3 - 5x^2 + xy - 4y^2 + 2$
take $8x^3 - 7x^2y + xy^2 - y^3 + 9x^2 - xy + 6y^2 - 4$.
11. From $a^4 - b^4$ take $4a^3b - 6a^2b^2 + 4ab^3$, and from the
result take $2a^4 - 4a^3b + 6a^2b^2 + 4ab^3 - 2b^4$.
12. From $x^3y^2 - 3x^2y^2 + 4xy^4 - y^5$ take $-x^5 + 2x^4y - 4xy^4 - 4y^5$. Add the same two expressions, and subtract
the former result from the latter.
13. From $a^2b^2 - a^2bc - 8ab^2c - a^2c^2 + abc^2 - 6b^3c^2$
take $2a^2bc - 5ab^2c + 2abc^2 - 5b^3c^2$.

14. From $12a + 3b - 5c - 2d$ take $10a - b + 4c - 3d$, and show that the result is numerically correct when $a=6$, $b=4$, $c=1$, $d=5$.
15. What number must be added to a to make b ; and what number must be taken from $2a^3 - 6a^2b + 6ab^2 - 2b^3$ to leave $a^3 - 7a^2b - 3b^3$?
16. From $2x^2 - y^2 - 2xy + z^2$ take $x^2 - y^2 + 2xy - z^2$.
17. From $12ac + 8cd - 9$ take $-7ac - 9cd + 8$.
18. From $-6a^2 + 2ab - 3c^2$ take $4a^2 + 6ab - 4c^2$.
19. From $9xy - 4x - 3y + 7$ take $8xy - 2x + 3y + 6$.
20. From $-a^2bc - ab^2c + abc^2 - abc$
take $a^2bc + ab^2c - abc^2 + abc$.
21. From $7x^2 - 2x + 4$ take $2x^2 + 3x - 1$.
22. From $3x^2 + 2xy - y^2$ take $-x^2 - 3xy + 3y^2$, and from the remainder take $3x^2 + 4xy - 5y^2$.
23. From $ax^2 - by^2$ take $cx^2 - dy^2$.
24. From $ax + bx + by + cy$ take $ax - bx - by + cy$.
25. From $5x^2 + 4x - 4y + 3y^2$ take $5x^2 - 3x + 3y + y^2$.
26. From $a^2b^2 + 12abc - 9ax^2$ take $4ab^2 - 6acx + 3a^2x$.
27. From $a^2 - 2ab + c^2 - 3b^2$ take $2a^2 - 2ab + 3b^2$.
28. From the sum of the first four of the following expressions, $a^2 + b^2 + c^2 + d^2$, $d^2 + b^2 + c^2$, $a^2 - c^2 + b^2 - d^2$, $a^2 - b^2 + c^2 + d^2$, $b^2 + c^2 + d^2 - a^2$ take the sum of the last four.
29. From $2x^2 - 2y^2 - z^2$ take $3y^2 + 2x^2 - z^2$, and from the remainder take $3z^2 - 2y^2 - x^2$.
30. From $a^3 - 2a^2c + 3ac^2$ take the sum of $a^3c - 2a^3 + 2ac^3$ and $a^3 - ac^2 - a^2c$.

PARENTHESES.

54. From (§ 52), it appears that

$$(1) \ a + (+b) = a + b$$

$$(2) \ a + (-b) = a - b.$$

$$(3) \ a - (+b) = a - b.$$

$$(4) \ a - (-b) = a + b.$$

The same laws respecting the removal of parentheses hold true whether one or more terms are inclosed. Hence, when an expression within a parenthesis is preceded by a **plus sign**, the parenthesis may be removed.

When an expression within a parenthesis is preceded by a **minus sign**, the parenthesis may be removed *if the sign of every term within the parenthesis be changed*. Thus :

$$(1) \ a + (b - c) = a + b - c.$$

$$(2) \ a - (b - c) = a - b + c.$$

55. Expressions may occur with more than one parenthesis. These parentheses may be removed in succession by removing *first, the innermost parenthesis*; next, the innermost of all that remain, and so on. Thus :

$$\begin{aligned} (1) \ a - \{b - (c - d)\} \\ &= a - \{b - c + d\}, \\ &= a - b + c - d. \end{aligned}$$

$$\begin{aligned} (2) \ a - [b - \{c + (d - e - f)\}] \\ &= a - [b - \{c + (d - e + f)\}], \\ &= a - [b - \{c + d - e + f\}], \\ &= a - [b - c - d + e - f], \\ &= a - b + c + d - e + f. \end{aligned}$$

EXERCISE 9.

Simplify the following expressions by removing the parentheses and combining like terms :

1. $(a+b) + (b+c) - (a+c)$.
2. $(2a-b-c) - (a-2b+c)$.
3. $(2x-y) - (2y-z) - (2z-x)$.
4. $(a-x-y) - (b-x+y) + (c+2y)$.
5. $(2x-y+3z) + (-x-y-4z) - (3x-2y-z)$.
6. $(3a-b+7c) - (2a+3b) - (5b-4c) + (3c-a)$.
7. $1 - (1-a) + (1-a+a^2) - (1-a+a^2-a^3)$.
8. $a - \{2b - (3c+2b) - a\}$.
9. $2a - \{b - (a-2b)\}$.
10. $3a - \{b + (2a-b) - (a-b)\}$.
11. $7a - [3a - \{4a - (5a-2a)\}]$.
12. $2x + (y-3z) - \{(3x-2y) + z\} + 5x - (4y-3z)$.
13. $\{(3a-2b) + (4c-a)\} - \{a - (2b-3a) - c\}$
 $\quad\quad\quad + \{a - (b-5c-a)\}$.
14. $a - [2a + (3a-4a)] - 5a - \{6a - [(7a+8a) - 9a]\}$.
15. $2a - (3b+2c) - [5b - (6c-6b) + 5c]$
 $\quad\quad\quad - \{2a - (c+2b)\}$.
16. $a - [2b + \{3c - 3a - (a+b)\} + \{2a - (b+c)\}]$.
17. $16 - x - [7x - \{8x - (9x - \overline{3x-6x})\}]$.
18. $2a - [3b + (2b-c) - 4c + \{2a - (3b - \overline{c-2b})\}]$.
19. $a - [2b + \{3c - 3a - (a+b)\} + 2a - (b+3c)]$.
20. $a - [5b - \{a - (3c-3b) + 2c - (a-2b-c)\}]$.

56. The rules for introducing parentheses follow directly from the rules for removing them :

1. Any number of terms of an expression may be put within a parenthesis, and the sign **plus** placed before the whole.

2. Any number of terms of an expression may be put within a parenthesis, and the sign **minus** placed before the whole ; *provided the sign of every term within the parenthesis be changed.*

It is usual to prefix to the parenthesis the sign of the first term that is to be inclosed within it.

EXERCISE 10.

Express in binomials, and also in trinomials :

1. $2a - 3b - 4c + d + 3e - 2f$.
2. $a - 2x + 4y - 3z - 2b + c$.
3. $a^5 + 3a^4 - 2a^3 - 4a^2 + a - 1$.
4. $-3a - 2b + 2c - 5d - e - 2f$.
5. $ax - by - cz - bx + cy + az$.
6. $2x^5 - 3x^4y + 4x^3y^2 - 5x^2y^3 + xy^4 - 2y^5$.
7. Express each of the above in trinomials, having the last two terms inclosed by *inner* parentheses.

Collect in parentheses the coefficients of x , y , z , in

8. $2ax - 6ay + 4bz - 4bx - 2cx - 3cy$.
9. $ax - bx + 2ay + 3y + 4az - 3bz - 2z$.
10. $ax - 2by + 5cz - 4bx - 3cy + az - 2cx - ay + 4bz$.
11. $12ax + 12ay + 4by - 12bz - 15cx + 6cy + 3cz$.
12. $2ax - 3by - 7cz - 2bx + 2cx + 8cy - 2cx - cy - cz$.

CHAPTER III.

MULTIPLICATION OF ALGEBRAIC NUMBERS.

57. THE operation of finding the sum of 3 numbers, each equal to 5, is symbolized by the expression, $3 \times 5 = 15$, read, "three times five is equal to fifteen"; or, by the expression $5 \times 3 = 15$, read, "five multiplied by three is equal to fifteen."

58. With reference to this operation, this sum is called the **product**; one of the equal numbers is called the **multiplicand**; and the number which shows how many times the multiplicand is to be taken is called the **multiplier**.

59. The multiplier means so many **times**. The multiplicand can be a *positive* or a *negative number*; but the multiplier, when integral, means that the multiplicand is taken *so many times to be added, or to be subtracted*.

60. If we have to multiply 867 by 98, we may put the multiplier in the form $100 - 2$. The 100 will mean that the multiplicand is taken 100 times *to be added*; the -2 will mean that the multiplicand is taken twice *to be subtracted*.

In general, an integral multiplier with $+$ before it, expressed or understood, means that the multiplicand is taken so many times *to be added*; and a multiplier with $-$ before it means that the multiplicand is taken so many times *to be subtracted*. Thus,

$$(1) +3 \times (+5) = (+5) + (+5) + (+5), \text{ or } (+15).$$

$$(2) +3 \times (-5) = (-5) + (-5) + (-5), \text{ or } (-15).$$

$$(3) -3 \times (+5) = -(+5) - (+5) - (+5), \text{ or } (-15).$$

$$(4) -3 \times (-5) = -(-5) - (-5) - (-5), \text{ or } (+15).$$

From these four cases it follows, that, in finding the product of two numbers,

61. *Like signs produce plus; unlike signs, minus.*

EXERCISE 11.

$$1. -17 \times 8 =$$

$$4. -18 \times -5 =$$

$$2. -12.8 \times 25 =$$

$$5. 43 \times -6 =$$

$$3. 3.29 \times 5.49 =$$

$$6. 457 \times 100 =$$

$$7. (-358 - 417) \times -79 =$$

$$8. (7.512 - \{-2.894\}) \times (-6.037 + \{13.963\}) =$$

62. The product of more than two factors, each preceded by $-$, will be positive or negative, according as the number of such factors is even or odd. Thus,

$$-2 \times -3 \times -4 = +6 \times -4 = -24.$$

$$-2 \times -3 \times -4 \times -5 = -24 \times -5 = +120.$$

$$9. 13 \times 8 \times -7 =$$

$$10. -38 \times 9 \times -6 =$$

$$11. -20.9 \times -1.1 \times 8 =$$

$$12. -78.3 \times -0.57 \times +1.38 \times -27.9 =$$

$$13. -2.906 \times -2.076 \times -1.49 \times 0.89 =$$

MULTIPLICATION OF MONOMIALS.

63. The product of numerical factors is a new number in which no trace of the original factors is found. Thus, $4 \times 9 = 36$. But the product of literal factors can only be expressed by writing them one after the other. Thus, the product of a and b is expressed by ab ; the product of ab and cd is expressed by $abcd$.

64. If we have to multiply $5a$ by $-4b$, the factors will give the same result in whatever order they are taken. Thus, $5a \times -4b = 5 \times -4 \times a \times b = -20 \times ab = -20ab$.

65. Hence, to find the product of monomials, *annex the literal factors to the product of the numerical factors.*

$$\mathbf{66.} \quad a^2 \times a^3 = aa \times aaa = aaaaa = a^5.$$

$$a^2 \times a^3 \times a^4 = aa \times aaa \times aaaa = aaaaaaaaa = a^9.$$

It is evident that the exponent of the product is equal to the sum of the exponents of the factors. Hence,

67. *The product of two or more powers of any number is that number with an exponent equal to the sum of the exponents of the factors.*

EXERCISE 12.

$$1. \quad +a \times +b = +ab.$$

$$6. \quad -3p \times 8m = -24pm.$$

$$2. \quad +a \times -b = -ab.$$

$$7. \quad 3a^2 \times -a^3 = -3a^5.$$

$$3. \quad -a \times +b = -ab.$$

$$8. \quad -3a \times 2a^5 = -6a^6.$$

$$4. \quad -a \times -b = +ab.$$

$$9. \quad 6a \times -2a =$$

$$5. \quad 7a \times 5b = 35ab.$$

$$10. \quad 5mn \times 9m =$$

11. $3ax \times -4by =$
12. $-8cm \times dn =$
13. $-7ab \times 2ac =$
14. $5m^2x \times 3mx^2 =$
15. $5a^m \times -2a^n =$
16. $3a^2x^2 \times 7a^3x^4 =$
17. $7a \times -4b \times -8c =$
18. $8ab^2 \times 3ac \times -4c^2 =$
19. $27ab \times -39mp \times 18ap =$
20. $6ab^2y^3 \times 2b^3y^2 \times -5a^3y =$
21. $7m^2x \times 3mx^2 \times -2mq =$
22. $-3pq^2 \times 6p^2q \times 8p^2q^2 =$
23. $2a^2m^3x^4 \times 3am^5x^2 \times 4a^3mx^2 =$
24. $6x^2yz^2 \times -9x^2y^2z^2 \times -3x^4yz =$
25. $3ax \times 2am \times -4mx \times b^2 =$
26. $7am^2 \times 3b^2n^2 \times -4ab \times a^2bn \times -2b^2n \times -mn^2 =$

OF POLYNOMIALS BY MONOMIALS.

68. If we have to multiply $a + b$ by n , that is, to take $(a + b)n$ times to be added, we have,

$$\begin{aligned}
 (a + b) \times n &= (a + b) + (a + b) + (a + b) \dots n \text{ times,} \\
 &= a + a + a \dots n \text{ times} + b + b + b \dots n \text{ times,} \\
 &= a \times n + b \times n, \\
 &= an + bn.
 \end{aligned}$$

As it is immaterial in what order the factors are taken,

$$n \times (a + b) = an + bn.$$

In like manner,

$$(a + b + c) \times n = an + bn + cn,$$

or,

$$n(a + b + c) = an + bn + cn.$$

Hence, to multiply a polynomial by a monomial,

69. *Multiply each term of the polynomial by the monomial, and add the partial products.*

EXERCISE 13.

1. $(6a - 5b) \times 3c = 18ac - 15bc.$
2. $(2 + 3a - 4a^2 - 5a^3)6a^2 = 12a^2 + 18a^3 - 24a^4 - 30a^5.$
3. $5a(3b + 4c - d) = 15ab + 20ac - 5ad.$
4. $-3ax(-by - 2cz + 5) = 3abxy + 6acxz - 15ax.$
5. $(4a^2 - 3b) \times 3ab =$
6. $(8a^2 - 9ab) \times 3a^2 =$
7. $(3x^2 - 4y^2 + 5z^2) \times 2x^2y =$
8. $(a^3x - 5a^2x^2 + ax^3 + 2x^4) \times ax^2y =$
9. $(-9a^5 + 3a^3b^2 - 4a^2b^3 - b^5) \times -3ab^4 =$
10. $(3x^3 - 2x^2y - 7xy^2 + y^3) \times -5x^2y =$
11. $(-4xy^2 + 5x^2y + 8x^3) \times -3x^2y =$
12. $(-3 + 2ab + a^2b^2) \times -a^4 =$
13. $(-z - 2xz^2 + 5x^2yz^2 - 6x^3y^2 + 3x^3y^2z) \times -3x^3yz =$

OF POLYNOMIALS BY POLYNOMIALS.

70. If we have $a + b + c$ to be multiplied by $m + n + p$, we may represent the multiplicand $a + b + c$ by M . Then

$$M(m + n + p) = M \times m + M \times n + M \times p.$$

If now we substitute for M its value,

$$\begin{aligned} (a + b + c)(m + n + p) &= (a + b + c) \times m \\ &\quad + (a + b + c) \times n \\ &\quad + (a + b + c) \times p; \end{aligned}$$

$$\text{or, } (a + b + c)(m + n + p) = am + bm + cm \\ + an + bn + cn \\ + ap + bp + cp.$$

That is, to find the product of two polynomials,

71. *Multiply the multiplicand by each term of the multiplier and add the partial products; or, multiply each term of one factor by each term of the other and add the partial products.*

72. In multiplying polynomials, it is a convenient arrangement to write the multiplier under the multiplicand, and place like terms of the partial products in columns.

$$\text{Thus: (1) } \begin{array}{r} 5a - 6b \\ 3a - 4b \\ \hline 15a^2 - 18ab \\ -20ab + 24b^2 \\ \hline 15a^2 - 38ab + 24b^2 \end{array}$$

(2) Multiply $4x + 3 + 5x^2 - 6x^3$ by $4 - 6x^2 - 5x$.

Arrange both multiplicand and multiplier according to the ascending powers of x .

$$\begin{array}{r} 3 + 4x + 5x^2 - 6x^3 \\ 4 - 5x - 6x^2 \\ \hline 12 + 16x + 20x^2 - 24x^3 \\ -15x - 20x^2 - 25x^3 + 30x^4 \\ -18x^2 - 24x^3 - 30x^4 + 36x^5 \\ \hline 12 + x - 18x^2 - 73x^3 + 36x^5 \end{array}$$

(3) Multiply $1 + 2x + x^4 - 3x^3$ by $x^3 - 2 - 2x$.

Arrange according to the descending powers of x .

$$\begin{array}{r} x^4 - 3x^3 + 2x + 1 \\ x^3 - 2x - 2 \\ \hline x^7 - 3x^6 + 2x^4 + x^3 \\ -2x^5 + 6x^3 - 4x^2 - 2x \\ -2x^4 + 6x^2 - 4x - 2 \\ \hline x^7 - 5x^5 + 7x^3 + 2x^2 - 6x - 2 \end{array}$$

(4) Multiply $a^3 + b^3 + c^3 - ab - bc - ac$ by $a + b + c$.

Arrange according to the descending powers of a .

$$\begin{array}{r}
 a^3 - ab - ac + b^3 - bc + c^3 \\
 a + b + c \\
 \hline
 a^3 - a^2b - a^2c + ab^2 - abc + ac^3 \\
 + a^2b \quad - ab^2 - abc \quad + b^3 - b^2c + bc^2 \\
 \hline
 \quad + a^2c \quad - abc - ac^2 \quad + b^2c - bc^2 + c^3 \\
 \hline
 a^3 \quad \quad - 3abc \quad + b^3 \quad \quad + c^3
 \end{array}$$

The student should observe that, with a view to bringing like terms of the partial products in columns, the terms of the multiplicand and multiplier are arranged in the **same order**.

In order to test the accuracy of the work, interchange the multiplicand and multiplier. The result should be the same in both operations.

EXERCISE 14.

Multiply :

1. $x^2 - 4$ by $x^2 + 5$.
2. $y - 6$ by $y + 13$.
3. $a^4 + a^2x^2 + x^4$ by $a^2 - x^2$.
4. $x^2 + xy + y^2$ by $x - y$.
5. $2x - y$ by $x + 2y$.
6. $2x^3 + 4x^2 + 8x + 16$ by $3x - 6$.
7. $x^3 + x^2 + x - 1$ by $x - 1$.
8. $x^2 - 3ax$ by $x + 3a$.
9. $2b^2 + 3ab - a^2$ by $-5b + 7a$.
10. $2a + b$ by $a + 2b$.
11. $a^3 + ab + b^2$ by $a - b$.
12. $a^2 - ab + b^2$ by $a + b$.
13. $2ab - 5b^2$ by $3a^2 - 4ab$.
14. $-a^3 + 2a^2b - b^3$ by $4a^2 + 8ab$.
15. $a^3 + ab + b^2$ by $a^2 - ab + b^2$.

16. $a^3 - 3a^2b + 3ab^2 - b^3$ by $a^2 - 2ab + b^2$.
17. $x + 2y - 3z$ by $x - 2y + 3z$.
18. $2x^2 + 3xy + 4y^2$ by $3x^2 - 4xy + yz$.
19. $x^2 + xy + y^2$ by $x^2 + xz + z^2$.
20. $a^2 + b^2 + c^2 - ab - ac - bc$ by $a + b + c$.
21. $x^2 - xy + y^2 + x + y + 1$ by $x + y - 1$.

Arrange the multiplicand and multiplier according to the descending powers of a common letter, and multiply :

22. $5x + 4x^2 + x^3 - 24$ by $x^2 + 11 - 4x$.
23. $x^3 + 11x - 4x^2 - 24$ by $x^2 + 5 + 4x$.
24. $x^4 + x^3 - 4x - 11 + 2x^2$ by $x^2 - 2x + 3$.
25. $-5x^4 - x^2 - x + x^5 + 13x^3$ by $x^2 - 2 - 2x$.
26. $3x + x^3 - 2x^2 - 4$ by $2x + 4x^2 + 3x^3 + 1$.
27. $5a^4 + 2a^2b^2 + ab^3 - 3a^3b$ by $5a^3b - 2ab^3 + 3a^2b^2 + b^4$.
28. $4a^7y - 32ay^4 - 8a^5y^3 + 16a^3y^5$ by $a^6y^2 + 4a^2y^4 + 4a^4y^3$.
29. $3m^3 + 3n^3 + 9mn^2 + 9m^2n$ by $6m^2n^3 - 2mn^4$
 $- 6m^3n^2 + 2m^4n$.
30. $6a^5b + 3a^2b^4 - 2ab^5 + b^6$ by $4a^4 - 2ab^3 - 3b^4$.

Find the products of :

31. $x - 3$, $x - 1$, $x + 1$, and $x + 3$.
32. $x^2 - x + 1$, $x^2 + x + 1$, and $x^4 - x^2 + 1$.
33. $a^2 + ab + b^2$, $a^2 - ab + b^2$, and $a^4 - a^2b^2 + b^4$.
34. $4a^3 - 4a^2b + ab^2$, $4a^2 + 3ab + b^2$, and $2a^2b + b^3$.
35. $x + a$, $x + 2a$, $x - 3a$, $x - 4a$, and $x + 5a$.
36. $9a^2 + b^2$, $27a^3 - b^3$, $27a^5 + b^3$, and $81a^4 - 9a^2b^2 + b^4$.

37. From the product of $y^2 - 2yz - z^2$ and $y^2 + 2yz - z^2$ take the product of $y^2 - yz - 2z^2$ and $y^2 + yz - 2z^2$.
38. Find the dividend when the divisor $= 3a^2 - ab - 3b^2$, the quotient $= a^2b - 2b^3$ the remainder $= -2ab^4 - 6b^5$.

The multiplication of polynomials may be *indicated* by inclosing each in a parenthesis and writing them one after the other. When the operations indicated are actually performed, the expression is said to be *simplified*.

Simplify :

39. $(a+b-c)(a+c-b)(b+c-a)(a+b+c)$.
40. $(a+b)(b+c) - (c+d)(d+a) - (a+c)(b-d)$.
41. $(a+b+c+d)^2 + (a-b-c+d)^2$
 $+ (a-b+c-d)^2 + (a+b-c-d)^2$.
42. $(a+b+c)^2 - a(b+c-a) - b(a+c-b) - c(a+b-c)$.
43. $(a-b)x - (b-c)a - \{(b-x)(b-a) - (b-c)(b+c)\}$.
44. $(m+n)m - \{(m-n)^2 - (n-m)n\}$.
45. $(a-b+c)^2 - \{a(c-a-b) - [b(a+b+c) - c(a-b-c)]\}$.
46. $(p^2+q^2)r - (p+q)(p\{r-q\} - q\{r-p\})$.
47. $(9x^2y^2 - 4y^4)(x^2 - y^2) - \{3xy - 2y^2\}\{3x(x^2+y^2) - 2y(y^2+3xy-x^2)\}y$.
48. $a^2 - \{2ab - [-(a+\{b-c\})(a-\{b-c\}) + 2ab] - 4bc\} - (b+c)^2$.
49. $\{ac - (a-b)(b+c)\} - b\{b - (a-c)\}$.
50. $5\{(a-b)x - cy\} - 2\{a(x-y) - bx\} - \{3ax - (5c-2a)y\}$.
51. $(x-1)(x-2) - 3x(x+3) + 2\{(x+2)(x+1) - 3\}$.

52. $\{(2a+b)^2 + (a-2b)^2\} \times \{(3a-2b)^2 - (2a-3b)^2\}$.
 53. $4(a-3b)(a+3b) - 2(a-6b)^2 - 2(a^2+6b^2)$.
 54. $x^2(x^2+y^2)^2 - 2x^2y^2(x+y)(x-y) - (x^3-y^3)^2$.
 55. $16(a^2+b^2)(a^2-b^2) - (2a-3)(2a+3)(4a^2+9)$
 $\quad + (2b-3)(2b+3)(4b^2+9)$.
-

73. There are some examples in multiplication which occur so often in algebraical operations that they should be carefully noticed and remembered. The three which follow are of great importance :

(1) $\begin{array}{r} a + b \\ a + b \\ \hline a^2 + ab \\ \quad ab + b^2 \\ \hline a^2 + 2ab + b^2 \end{array}$	(2) $\begin{array}{r} a - b \\ a - b \\ \hline a^2 - ab \\ \quad - ab + b^2 \\ \hline a^2 - 2ab + b^2 \end{array}$	(3) $\begin{array}{r} a + b \\ a - b \\ \hline a^2 + ab \\ \quad - ab - b^2 \\ \hline a^2 - b^2 \end{array}$
--	--	--

From (1) we have $(a+b)^2 = a^2 + 2ab + b^2$. That is,

74. *The square of the sum of two numbers is equal to the sum of their squares + twice their product.*

From (2) we have $(a-b)^2 = a^2 - 2ab + b^2$. That is,

75. *The square of the difference of two numbers is equal to the sum of their squares - twice their product.*

From (3) we have $(a+b)(a-b) = a^2 - b^2$. That is,

76. *The product of the sum and difference of two numbers is equal to the difference of their squares.*

77. A general truth expressed by symbols is called a **formula**.

78. By using the double sign \pm , read plus or minus, we may represent (1) and (2) by a single formula; thus,

$$(a \pm b)^2 = a^2 \pm 2ab + b^2;$$

in which expression the upper signs correspond with each other, and the lower signs with each other.

By remembering these formulas the square of any binomial, or the product of the sum and difference of any two numbers, may be written by inspection; thus :

EXERCISE 15.

1. $(127)^2 - (123)^2 = (127 + 123)(127 - 123)$
 $= 250 \times 4 = 1000.$
 2. $(29)^2 = (30 - 1)^2 = 900 - 60 + 1 = 841.$
 3. $(53)^2 = (50 + 3)^2 = 2500 + 300 + 9 = 2809.$
 4. $(3x + 2y)^2 = 9x^2 + 12xy + 4y^2.$
 5. $(2a^2x - 5x^2y)^2 = 4a^4x^2 - 20a^2x^3y + 25x^4y^2.$
 6. $(3ab^2c + 2a^2c^2)(3ab^2c - 2a^2c^2) = 9a^2b^4c^2 - 4a^4c^4.$
- | | |
|------------------------|--------------------------|
| 7. $(x + y)^2 =$ | 15. $(ab + cd)^2 =$ |
| 8. $(y - z)^2 =$ | 16. $(3mn - 4)^2 =$ |
| 9. $(2x + 1)^2 =$ | 17. $(12 + 5x)^2 =$ |
| 10. $(2a + 5b)^2 =$ | 18. $(4xy^2 - yz^2)^2 =$ |
| 11. $(1 - x^2)^2 =$ | 19. $(3abc - bcd)^2 =$ |
| 12. $(3ax - 4x^2)^2 =$ | 20. $(4x^3 - xy^2)^2 =$ |
| 13. $(1 - 7a)^2 =$ | 21. $(x + y)(x - y) =$ |
| 14. $(5xy + 2)^2 =$ | 22. $(2a + b)(2a - b) =$ |

$$23. (3-x)(3+x)=$$

$$24. (3ab+2b^2)(3ab-2b^2)=$$

$$25. (4x^2-3y^2)(4x^2+3y^2)=$$

$$26. (a^3x^2-by^4)(a^3x^2+by^4)=$$

$$27. (6xy-5y^2)(6xy+5y^2)=$$

$$28. (4x^5-1)(4x^5+1)=$$

$$29. (1+3ab^2)(1-3ab^2)=$$

$$30. (ax+by)(ax-by)(a^2x^2+b^2y^2)=$$

79. Also the square of a trinomial should be carefully noticed.

$$\begin{array}{r} a + b + c \\ a + b + c \\ \hline a^2 + ab + ac \\ \quad ab \quad + b^2 + bc \\ \quad \quad ac \quad + bc + c^2 \\ \hline a^2 + 2ab + 2ac + b^2 + 2bc + c^2, \\ = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc. \end{array}$$

It is evident that this result is composed of two sets of numbers :

I. The squares of a , b , and c ;

II. Twice the products of a , b , and c taken two and two.

$$\begin{array}{r} \text{Again, } a - b - c \\ a - b - c \\ \hline a^2 - ab - ac \\ \quad - ab \quad + b^2 + bc \\ \quad \quad - ac \quad + bc + c^2 \\ \hline a^2 - 2ab - 2ac + b^2 + 2bc + c^2 \\ = a^2 + b^2 + c^2 - 2ab - 2ac + 2bc. \end{array}$$

The law of formation is the same as before :

- I. The squares of a , b , and c ;
- II. Twice the products of a , b , and c taken two and two.

The sign of each double product is $+$ or $-$ according as the signs of the factors composing it are *like* or *unlike*.

The same law holds good for the square of expressions containing more than three terms, and may be stated thus :

80. *To the sum of the squares of the several terms add twice the product of each term by each of the terms that follow it.*

By remembering this formula, the square of any polynomial may be written by inspection; thus :

EXERCISE 16.

- | | |
|--------------------------------|-------------------------------|
| 1. $(x + y + z)^2 =$ | 9. $(a^3 + b^3 + c^3)^2 =$ |
| 2. $(x - y + z)^2 =$ | 10. $(x^3 - y^3 - z^3)^2 =$ |
| 3. $(m + n - p - q)^2 =$ | 11. $(x + 2y - 3z)^2 =$ |
| 4. $(x^2 + 2x - 3)^2 =$ | 12. $(x^2 - 2y^2 + 5z^2)^2 =$ |
| 5. $(x^2 - 6x + 7)^2 =$ | 13. $(x^2 + 2x - 2)^2 =$ |
| 6. $(2x^2 - 7x + 9)^2 =$ | 14. $(x^2 - 5x + 7)^2 =$ |
| 7. $(x^2 + y^2 - z^2)^2 =$ | 15. $(2x^2 - 3x - 4)^2 =$ |
| 8. $(x^4 - 4x^2y^2 + y^4)^2 =$ | 16. $(x + 2y + 3z)^2 =$ |

81. Likewise, the product of two binomials of the form $x + a$, $x + b$ should be carefully noticed and remembered.

$$\begin{array}{r}
 (1) \quad x + 5 \\
 \underline{x + 3} \\
 x^2 + 5x \\
 \quad + 3x + 15 \\
 \hline
 x^2 + 8x + 15
 \end{array}$$

$$\begin{array}{r}
 (2) \quad x - 5 \\
 \underline{x - 3} \\
 x^2 - 5x \\
 \quad - 3x + 15 \\
 \hline
 x^2 - 8x + 15
 \end{array}$$

(3) $x + 5$

$x - 3$

$x^2 + 5x$

$- 3x - 15$

$x^2 + 2x - 15$

(4) $x - 5$

$x + 3$

$x^2 - 5x$

$+ 3x - 15$

$x^2 - 2x - 15$

It will be observed that:

I. In all the results the first term is x^2 and the last term is the product of 5 and 3.

II. From (1) and (2), when the second terms of the binomials have *like* signs, the product has

the last term *positive*;

the *coefficient* of the middle term = the **sum** of 3 and 5;

the *sign* of the middle term is the same as that of the 3 and 5.

III. From (3) and (4), when the second terms of the binomials have *unlike* signs, the product has

the last term *negative*;

the *coefficient* of the middle term = the **difference** of 3 and 5;

the *sign* of the middle term is that of the **greater** of the two numbers.

82. These results may be deduced from the general formula,

$$(x + a)(x + b) = x^2 + (a + b)x + ab,$$

by supposing for (1) a and b both positive;

(2) a and b both negative;

(3) a positive, b negative, and $a > b$;

(4) a negative, b positive, and $a > b$.

By remembering this formula the product of two binomials may be written by inspection; thus:

EXERCISE 17.

- | | |
|---------------------|----------------------------|
| 1. $(x+2)(x+3)=$ | 11. $(x-c)(x-d)=$ |
| 2. $(x+1)(x+5)=$ | 12. $(x-4y)(x+y)=$ |
| 3. $(x-3)(x-6)=$ | 13. $(a-2b)(a-5b)=$ |
| 4. $(x-8)(x-1)=$ | 14. $(x^2+2y^2)(x^2+y^2)=$ |
| 5. $(x-8)(x+1)=$ | 15. $(x^2-3xy)(x^2+xy)=$ |
| 6. $(x-2)(x+5)=$ | 16. $(ax-9)(ax+6)=$ |
| 7. $(x-3)(x+7)=$ | 17. $(x+a)(x-b)=$ |
| 8. $(x-2)(x-4)=$ | 18. $(x-11)(x+4)=$ |
| 9. $(x+1)(x+11)=$ | 19. $(x+12)(x-11)=$ |
| 10. $(x-2a)(x+3a)=$ | 20. $(x-10)(x-5)=$ |

83. The second, third, and fourth powers of $a+b$ are found in the following manner:

$$\begin{array}{r}
 a + b \\
 a + b \\
 \hline
 a^2 + ab \\
 \quad ab + b^2 \\
 \hline
 (a+b)^2 = a^2 + 2ab + b^2 \\
 \quad a + b \\
 \quad a^2 + 2a^2b + ab^2 \\
 \quad \quad a^2b + 2ab^2 + b^3 \\
 \hline
 (a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 \\
 \quad a + b \\
 \quad a^4 + 3a^3b + 3a^2b^2 + ab^3 \\
 \quad \quad a^3b + 3a^2b^2 + 3ab^3 + b^4 \\
 \hline
 (a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4
 \end{array}$$

From these results it will be observed that :

I. The number of terms is greater by one than the exponent of the power to which the binomial is raised.

II. In the first term, the exponent of a is the same as the exponent of the power to which the binomial is raised; and it decreases by one in each succeeding term.

III. b appears in the second term with one for an exponent, and its exponent increases by one in each succeeding term.

IV. The coefficient of the first term is 1.

V. The coefficient of the second term is the same as the exponent of the power to which the binomial is raised.

VI. The coefficient of each succeeding term is found from the next preceding term by multiplying its coefficient by the exponent of a , and dividing the product by a number greater by one than the exponent of b .

84. If b is negative, the terms in which the odd powers of b occur are negative. Thus :

$$(a - b)^4 = a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4.$$

EXERCISE 18.

Write by inspection the results :

- | | | |
|------------------|------------------|-------------------|
| 1. $(x + a)^3 =$ | 5. $(x + a)^4 =$ | 9. $(x + y)^5 =$ |
| 2. $(x - a)^3 =$ | 6. $(x - a)^4 =$ | 10. $(x - y)^5 =$ |
| 3. $(x + 1)^3 =$ | 7. $(x + 1)^4 =$ | 11. $(x + 1)^5 =$ |
| 4. $(x - 1)^3 =$ | 8. $(x - 1)^4 =$ | 12. $(x - 1)^5 =$ |

CHAPTER IV.

DIVISION.

85. Division is the operation by which, when a product and one of its factors are given, the other factor is determined.

86. With reference to this operation the product is called the **dividend**; the given factor the **divisor**; and the required factor the **quotient**.

87. The operation of division is indicated by the sign \div ; by the colon :, or by writing the dividend over the divisor with a line drawn between them. Thus, $12 \div 4$, $12 : 4$, $\frac{12}{4}$, each means that 12 is to be divided by 4.

88. $+12$ divided by $+4$ gives the quotient $+3$; since only a positive number, $+3$, when multiplied by $+4$, can give the positive product, $+12$. § 61.

$+12$ divided by -4 gives the quotient -3 ; since only a negative number, -3 , when multiplied by -4 , can give the positive product, $+12$. § 61.

-12 divided by $+4$ gives the quotient -3 ; since only a negative number, -3 , when multiplied by $+4$, can give the negative product, -12 . § 61.

-12 divided by -4 gives the quotient $+3$; since only a positive number, $+3$, when multiplied by -4 , can give the negative product, -12 . § 61.

$$(1) \frac{+12}{+4} = +3.$$

$$(3) \frac{-12}{+4} = -3,$$

$$(2) \frac{+12}{-4} = -3.$$

$$(4) \frac{-12}{-4} = +3.$$

From (1) and (4) it follows that

89. The quotient is **positive** when the dividend and divisor have *like* signs.

From (2) and (3) it follows that

The quotient is **negative** when the dividend and divisor have *unlike* signs.

90. The **absolute value** of the quotient is equal to the quotient of the **absolute values** of the dividend and divisor.

EXERCISE 19.

$$1. \frac{+264}{+4} =$$

$$3. \frac{+3840}{-30} =$$

$$5. \frac{106.33}{-4.9} =$$

$$2. \frac{-4648}{-8} =$$

$$4. \frac{-2568}{+12} =$$

$$6. \frac{-42.435}{+34.5} =$$

$$7. \frac{-264}{+24} =$$

$$10. \frac{-7.1560}{+324} =$$

$$8. \frac{-3670}{-85} =$$

$$11. \frac{-1}{-3.14159} =$$

$$9. \frac{+6.8503}{-61} =$$

$$12. \frac{-0.31831}{-31.4159} =$$

DIVISION OF MONOMIALS.

91. If we have to divide abc by bc , $aabx$ by aby , $12abc$ by $-4ab$, we write them as follows:

$$\frac{abc}{bc} = a. \quad \frac{aax}{aby} = \frac{ax}{y}. \quad \frac{12abc}{-4ab} = -3c.$$

Hence, to divide one monomial by another,

92. *Write the dividend over the divisor with a line between them; if the expressions have common factors, remove the common factors.*

If we have to divide a^5 by a^2 , a^6 by a^4 , a^4 by a , we write them as follows:

$$\frac{a^5}{a^2} = \frac{aaaaa}{aa} = aaa = a^3,$$

$$\frac{a^6}{a^4} = \frac{aaaaaa}{aaaa} = aa = a^2,$$

$$\frac{a^4}{a} = \frac{aaaa}{a} = aaa = a^3.$$

93. That is, if a power of a number is divided by a lower power of the same number, the quotient is that power of the number whose exponent is equal to the exponent of the dividend minus that of the divisor.

Again,

$$\frac{a^2}{a^5} = \frac{aa}{aaaaa} = \frac{1}{aaa} = \frac{1}{a^3},$$

$$\frac{a^3}{a^5} = \frac{aaa}{aaaaa} = \frac{1}{aa} = \frac{1}{a^2},$$

$$\frac{a^4}{a^8} = \frac{aaaa}{aaaaaaaa} = \frac{1}{aaaa} = \frac{1}{a^4}.$$

94. That is, if any power of a number is divided by a higher power of the same number, the quotient is expressed by 1 divided by the number with an exponent equal to the exponent of the divisor minus that of the dividend.

EXERCISE 20.

$$1. \frac{+ab}{+a} = +b. \quad 7. \frac{10ab}{2bc} = \quad 13. \frac{-3bm x}{4ax^2} =$$

$$2. \frac{+ab}{-a} = -b. \quad 8. \frac{x^3}{-x^5} = \quad 14. \frac{ab^2c^3}{abc} =$$

$$3. \frac{-ab}{+a} = -b. \quad 9. \frac{-12am}{-2m} = \quad 15. \frac{m^5p^2x^4}{mp^2x^2} =$$

$$4. \frac{-ab}{-a} = +b. \quad 10. \frac{35abcd}{5bd} = \quad 16. \frac{-51abdy^2}{3bdy} =$$

$$5. \frac{6mx}{2x} = \quad 11. \frac{abx}{5aby} = \quad 17. \frac{225m^3y}{25my^2} =$$

$$6. \frac{12a^4}{-3a} = \quad 12. \frac{27a^7}{-3a^3} = \quad 18. \frac{30x^2y^3}{-5x^3y} =$$

$$19. \frac{4a^2m^4x^5}{5a^5m^3x} = \quad 21. \frac{-3a^2b^3c^4d^5}{-a^4b^2cd^3} =$$

$$20. \frac{42x^3y^2z^4}{7xy^2z^3} = \quad 22. \frac{12am^5n^4p^3q^2}{4m^2n^3p^4q^5} =$$

$$23. (4a^2b^2z^3 \times 10a^2b^3z) \div 5a^2b^2z^2 =$$

$$24. (21x^2y^4z^6 \div 3xy^2z)(-2x^3y^2z) =$$

$$25. 104ab^2x^9 \div (91a^5b^6x^7 \div 7a^4b^4x) =$$

$$26. (24a^5b^3x \div 3a^2b^2) + (35a^6b^2x^2 \div -5a^3bx) =$$

$$27. 85a^{4m+1} \div 5a^{4m-2} = \quad 28. 84a^{n-4} \div 12a^2 =$$

OF POLYNOMIALS BY MONOMIALS.

95. The product of $(a + b + c) \times p = ap + bp + cp$.

If the product of two factors is divided by one of the factors, the quotient is the other factor. Therefore,

$$(ap + bp + cp) \div p = a + b + c.$$

But a , b , and c are the quotients obtained by dividing each term, ap , bp , and cp by p .

Therefore, to divide a polynomial by a monomial,

96. *Divide each term of the polynomial by the monomial.*

EXERCISE 21.

1. $(8ab - 12ac) \div 4a = 2b - 3c$.
2. $(15am - 10bm + 20cm) \div -5m = -3a + 2b - 4c$.
3. $(18amy - 27bny + 36cpy) \div -9y =$
4. $(21ax - 18bx + 15cx) \div -3x =$
5. $(12x^5 - 8x^3 + 4x) \div 4x =$
6. $(3x^3 - 6x^5 + 9x^7 - 12x^9) \div 3x^2 =$
7. $(35m^3y + 28m^2y^2 - 14my^5) \div -.7my =$
8. $(4a^4b - 6a^3b^2 + 12a^2b^3) \div 2a^2b =$
9. $(12x^3y^3 - 15x^4y^2 - 24x^5y) \div -3x^2y =$
10. $(12x^5y^4 - 24x^4y^2 + 36x^3y^3 - 12x^2y^2) \div 12x^2y^2 =$
11. $(3a^4 - 2a^5b - a^6b^2) \div a^4 =$
12. $(3x^3yz^2 + 6x^4yz^3 - 15x^5y^2z^3 + 18x^6y^3z) \div -3x^3yz =$
13. $(-16a^3b^2c^5 + 8a^4b^2c^4 - 12a^5b^3c^3) \div -4a^2b^2c^2 =$

OF POLYNOMIALS BY POLYNOMIALS.

$$\begin{array}{lcl}
 \text{97. If the divisor (one factor)} & = & a + b + c, \\
 \text{and the quotient (other factor)} & = & \frac{n + p + q,}{} \\
 \text{then the dividend (product)} & = & \begin{cases} an + bn + cn \\ + ap + bp + cp \\ + aq + bq + cq. \end{cases}
 \end{array}$$

The first term of the dividend is an ; that is, the product of a , the first term of the divisor, by n , the first term of the quotient. The first term n of the quotient is therefore found by dividing an , the first term of the dividend, by a , the first term of the divisor.

If the partial product formed by multiplying the entire divisor by n is subtracted from the dividend, the first term of the remainder ap is the product of a , the first term of the divisor, by p , the second term of the quotient. That is, the second term of the quotient is obtained by dividing the first term of the remainder by the first term of the divisor. In like manner, the third term of the quotient is obtained by dividing the first term of the new remainder by the first term of the divisor, and so on.

Therefore, to divide one polynomial by another,

98. *Divide the first term of the dividend by the first term of the divisor.*

Write the result as the first term of the quotient.

Multiply all the terms of the divisor by the first term of the quotient.

Subtract the product from the dividend.

If there is a remainder, consider it as a new dividend and proceed as before.

99. It is of great importance to *arrange both dividend and divisor according to the ascending or descending powers of some common letter, and to keep this order throughout the operation.*

EXERCISE 22.

Divide

(1) $a^2 + 2ab + b^2$ by $a + b$;

$$\begin{array}{r} a^2 + 2ab + b^2 \overline{) a + b} \\ a^2 + \quad ab \quad \quad a + b \\ \hline \quad ab + b^2 \\ \quad ab + b^2 \\ \hline \end{array}$$

(2) $a^2 - b^2$ by $a + b$;

$$\begin{array}{r} a^2 - \quad b^2 \overline{) a + b} \\ a^2 + ab \quad a - b \\ \hline \quad -ab - b^2 \\ \quad -ab - b^2 \\ \hline \end{array}$$

(3) $a^2 - 2ab + b^2$ by $a - b$;

$$\begin{array}{r} a^2 - 2ab + b^2 \overline{) a - b} \\ a^2 - \quad ab \quad \quad a - b \\ \hline \quad -ab + b^2 \\ \quad -ab + b^2 \\ \hline \end{array}$$

(4) $4a^4x^2 - 4a^2x^4 + x^6 - a^6$ by $x^2 - a^2$;

$$\begin{array}{r} x^6 - 4a^2x^4 + 4a^4x^2 - a^6 \overline{) x^2 - a^2} \\ x^6 - \quad a^2x^4 \quad \quad x^4 - 3a^2x^2 + a^4 \\ \hline \quad -3a^2x^4 + 4a^4x^2 - a^6 \\ \quad -3a^2x^4 + 3a^4x^2 \\ \hline \quad \quad a^4x^2 - a^6 \\ \quad \quad a^4x^2 - a^6 \\ \hline \end{array}$$

(5) $22a^2b^2 + 15b^4 + 3a^4 - 10a^3b - 22ab^3$ by $a^2 + 3b^2 - 2ab$;

$$\begin{array}{r} 3a^4 - 10a^3b + 22a^2b^2 - 22ab^3 + 15b^4 \overline{) a^2 - 2ab + 3b^2} \\ 3a^4 - \quad 6a^3b + \quad 9a^2b^2 \quad \quad 3a^2 - 4ab + 5b^2 \\ \hline \quad -4a^3b + 13a^2b^2 - 22ab^3 \\ \quad -4a^3b + \quad 8a^2b^2 - 12ab^3 \\ \hline \quad \quad 5a^2b^2 - 10ab^3 + 15b^4 \\ \quad \quad 5a^2b^2 - 10ab^3 + 15b^4 \\ \hline \end{array}$$

Divide

6. $x^2 - 7x + 12$ by $x - 3$.
7. $x^2 + x - 72$ by $x + 9$.
8. $2x^3 - x^2 + 3x - 9$ by $2x - 3$.
9. $6x^3 + 14x^2 - 4x + 24$ by $2x + 6$.
10. $3x^3 + x + 9x^2 - 1$ by $3x - 1$.
11. $7x^3 + 58x - 24x^2 - 21$ by $7x - 3$.
12. $x^6 - 1$ by $x - 1$.
13. $a^3 - 2ab^2 + b^3$ by $a - b$.
14. $x^4 - 81y^4$ by $x - 3y$.
15. $x^5 - y^5$ by $x - y$.
16. $a^5 + 32b^5$ by $a + 2b$.
17. $2a^4 + 27ab^3 - 81b^4$ by $a + 3b$.
18. $x^4 + 11x^2 - 12x - 5x^3 + 6$ by $3 + x^2 - 3x$.
19. $x^4 - 9x^2 + x^3 - 16x - 4$ by $x^2 + 4 + 4x$.
20. $36 + x^4 - 13x^2$ by $6 + x^2 + 5x$.
21. $x^4 + 64$ by $x^2 + 4x + 8$.
22. $x^4 + x^2 + 57 - 35x - 24x^2$ by $x^2 - 3 + 2x$.
23. $1 - x - 3x^2 - x^3$ by $1 + 2x + x^2$.
24. $x^6 - 2x^3 + 1$ by $x^2 - 2x + 1$.
25. $a^4 + 2a^2b^2 + 9b^4$ by $a^2 - 2ab + 3b^2$.
26. $4x^5 - x^3 + 4x$ by $2 + 2x^2 + 3x$.
27. $a^5 - 243$ by $a - 3$.
28. $18x^4 + 82x^2 + 40 - 67x - 45x^3$ by $3x^2 + 5 - 4x$.
29. $x^4 - 6xy - 9x^2 - y^2$ by $x^2 + y + 3x$.

30. $x^4 + 9x^2y^2 - 6x^3y - 4y^4$ by $x^2 - 3xy + 2y^2$.
31. $x^4 + x^2y^2 + y^4$ by $x^2 - xy + y^2$.
32. $x^5 + x^3 + x^4y + y^3 - 2xy^2 - x^3y^2$ by $x^3 + x - y$.
33. $2x^3 - 3y^3 + xy - xz - 4yz - z^3$ by $2x + 3y + z$.
34. $12 + 82x^2 + 106x^4 - 70x^5 - 112x^3 - 38x$
by $3 - 5x + 7x^2$.
35. $x^5 + y^5$ by $x^4 - x^3y + x^2y^2 - xy^3 + y^4$.
36. $2x^4 + 2x^2y^2 - 2xy^3 - 7x^3y - y^4$ by $2x^2 + y^2 - xy$.
37. $16x^4 + 4x^2y^2 + y^4$ by $4x^2 - 2xy + y^2$.
38. $32a^5b + 8a^3b^3 - ab^5 - 4a^2b^4 - 56a^4b^2$
by $b^3 - 4a^2b + 6ab^2$.
39. $1 + 5x^3 - 6x^4$ by $1 - x + 3x^2$.
40. $1 - 52a^4b^4 - 51a^3b^3$ by $4a^2b^3 + 3ab - 1$.
41. $x^7y - xy^7$ by $x^3y + 2xy^3 - 2x^2y^2 - y^4$.
42. $x^6 + 15x^4y^2 + 15x^2y^4 + y^6 - 6x^5y - 6xy^5 - 20x^3y^3$
by $x^3 - 3x^2y + 3xy^2 - y^3$.
43. $a^7 + 2a^3b^4 - 2a^4b^3 - 2a^6b - 6a^2b^5 - 3ab^6$
by $a^3 - 2a^2b - ab^2$.
44. $81x^6y + 18x^2y^5 - 54x^5y^2 - 18x^3y^4 - 18xy^6 - 9y^7$
by $3x^4 + x^2y^2 + y^4$.
45. $a^4 + 2a^3b + 8a^2b^2 + 8ab^3 + 16b^4$ by $a^2 + 4b^2$.
46. $8y^6 - x^6 + 21x^3y^3 - 24xy^5$ by $3xy - x^2 - y^2$.
47. $16a^4 + 9b^4 + 8a^2b^2$ by $4a^2 + 3b^2 - 4ab$.
48. $a^3 + b^3 + c^3 - 3abc$ by $a + b + c$.
49. $a^3 + 8b^3 + c^3 - 6abc$ by $a^2 + 4b^2 + c^2 - ac - 2ab - 2bc$.
50. $a^3 + b^3 + c^3 + 3a^2b + 3ab^2$ by $a + b + c$.

100. The operation of division may be shortened in some cases by the use of parentheses. Thus:

$$\begin{array}{r}
 x^3 + (a+b+c)x^2 + (ab+ac+bc)x + abc \overline{) x + b} \\
 x^3 + (\quad + b \quad) x^2 \overline{) x^2 + (a+c)x + ac} \\
 \hline
 (a \quad + c) x^2 + (ab+ac+bc)x \\
 (a \quad + c) x^2 + (ab \quad + bc)x \\
 \hline
 acx \\
 acx \\
 \hline
 + abc \\
 + abc
 \end{array}$$

EXERCISE 23.

Divide

- $a^2(b+c) + b^2(a-c) + c^2(a-b) + abc$ by $a+b+c$.
- $x^3 - (a+b+c)x^2 + (ab+ac+bc)x - abc$
by $x^2 - (a+b)x + ab$.
- $x^3 - 2ax^2 + (a^2+ab-b^2)x - a^2b + ab^2$ by $x - a + b$.
- $x^4 - (a^2-b-c)x^2 - (b-c)ax + bc$ by $x^2 - ax + c$.
- $y^3 - (m+n+p)y^2 + (mn+mp+np)y - mnp$ by $y - p$.
- $x^4 + (5+a)x^3 - (4-5a+b)x^2 - (4a+5b)x + 4b$
by $x^2 + 5x - 4$.
- $x^4 - (a+b+c+d)x^3 + (ab+ac+ad+bc+bd+cd)x^2$
 $- (abc+abd+acd+bcd)x + abcd$
by $x^2 - (a+c)x + ac$.
- $x^5 - (m-c)x^4 + (n-cm+d)x^3 + (r+cn-dm)x^2$
 $+ (cr+dn)x + dr$ by $x^3 - mx^2 + nx + r$.
- $x^5 - mx^4 + nx^3 - nx^2 + mx - 1$ by $x - 1$.
- $(x+y)^3 + 3(x+y)^2z + 3(x+y)z^2 + z^3$
by $(x+y)^2 + 2(x+y)z + z^2$.

101. There are some cases in Division which occur so often in algebraic operations that they should be carefully noticed and remembered.

CASE I.

The student may easily verify the following results :

$$(1) \frac{a^3 - b^3}{a - b} = a^2 + ab + b^2.$$

$$(2) \frac{27a^3 - 8b^3}{3a - 2b} = 9a^2 + 6ab + 4b^2.$$

$$(3) \frac{a^5 - b^5}{a - b} = a^4 + a^3b + a^2b^2 + ab^3 + b^4.$$

$$(4) \frac{a^5 - 32b^5}{a - 2b} = a^4 + 2a^3b + 4a^2b^2 + 8ab^3 + 16b^4.$$

From these results it may be assumed that :

102. *The difference of two equal odd powers of any two numbers is divisible by the difference of the numbers.*

It will also be seen that :

I. The number of terms in the quotient is equal to the exponent of the powers.

II. The signs of the quotient are all positive.

III. The first term of the quotient is obtained, as usual, by dividing the first term of the dividend by the first term of the divisor.

IV. Each succeeding term of the quotient may be obtained by dividing the preceding term of the quotient by the first term of the divisor, and multiplying the result by the second term of the divisor (disregarding the sign).

EXERCISE 24.

Write by inspection the results in the following examples :

1. $(y^3 - 1) \div (y - 1)$.
2. $(b^3 - 125) \div (b - 5)$.
3. $(a^3 - 216) \div (a - 6)$.
4. $(x^3 - 343) \div (x - 7)$.
5. $(x^5 - y^5) \div (x - y)$.
6. $(a^5 - 1) \div (a - 1)$.
7. $(1 - 8x^3) \div (1 - 2x)$.
8. $(x^5 - 32b^5) \div (x - 2b)$.
9. $(8a^3x^3 - 1) \div (2ax - 1)$.
10. $(1 - 27x^3y^3) \div (1 - 3xy)$.
11. $(64a^3b^3 - 27x^3) \div (4ab - 3x)$.
12. $(243a^5 - 1) \div (3a - 1)$.
13. $(32a^5 - 243b^5) \div (2a - 3b)$.

CASE II.

- (1) $\frac{a^3 + b^3}{a + b} = a^2 - ab + b^2$.
- (2) $\frac{27x^3 + 8y^3}{3x + 2y} = 9x^2 - 6xy + 4y^2$.
- (3) $\frac{a^5 + b^5}{a + b} = a^4 - a^3b + a^2b^2 - ab^3 + b^4$.
- (4) $\frac{243x^5 + 32y^5}{3x + 2y} = 81x^4 - 54x^3y + 36x^2y^2 - 24xy^3 + 16y^4$.

From these results it may be assumed that :

103. *The sum of two equal odd powers of two numbers is divisible by the sum of the numbers.*

The quotient may be found as in Case I., but the signs are alternately plus and minus.

EXERCISE 25.

Write by inspection the results in the following examples :

1. $(x^3 + y^3) \div (x + y)$.
2. $(x^5 + y^5) \div (x + y)$.
3. $(1 + 8a^3) \div (1 + 2a)$.
4. $(27a^3 + b^3) \div (3a + b)$.
5. $(8a^3x^3 + 1) \div (2ax + 1)$.
6. $(x^3 + 27y^3) \div (x + 3y)$.
7. $(a^5 + 32b^5) \div (a + 2b)$.
8. $(512x^3y^3 + z^3) \div (8xy + z)$.
9. $(729a^3 + 216b^3) \div (9a + 6b)$.
10. $(64a^3 + 1000b^3) \div (4a + 10b)$.
11. $(64a^3b^3 + 27x^3) \div (4ab + 3x)$.
12. $(x^3 + 343) \div (x + 7)$.
13. $(27x^3y^3 + 8z^3) \div (3xy + 2z)$.
14. $(1024a^5 + 243b^5) \div (4a + 3b)$.

CASE III.

- (1) $\frac{x^3 - y^3}{x - y} = x^2 + xy + y^2$.
- (2) $\frac{x^4 - y^4}{x - y} = x^3 + x^2y + xy^2 + y^3$.
- (3) $\frac{x^3 - y^3}{x + y} = x^2 - xy + y^2$.
- (4) $\frac{x^4 - y^4}{x + y} = x^3 - x^2y + xy^2 - y^3$.

From these results it may be assumed that :

104. *The difference of two equal even powers of two numbers is divisible by the difference and also by the sum of the numbers.*

When the divisor is the difference of the numbers, the quotient is found as in Case I.

When the divisor is the sum of the numbers, the quotient is found as in Case II.

EXERCISE 26.

Write by inspection the results in the following examples :

1. $(x^4 - y^4) \div (x - y)$.
2. $(x^4 - y^4) \div (x + y)$.
3. $(a^6 - x^6) \div (a - x)$.
4. $(a^6 - x^6) \div (a + x)$.
5. $(x^4 - 81y^4) \div (x - 3y)$.
6. $(x^4 - 81y^4) \div (x + 3y)$.
7. $(16x^4 - 1) \div (2x - 1)$.
8. $(16x^4 - 1) \div (2x + 1)$.
9. $(81a^4x^4 - 1) \div (3ax - 1)$.
10. $(81a^4x^4 - 1) \div (3ax + 1)$.
11. $(64a^6 - b^6) \div (2a - b)$.
12. $(64a^6 - b^6) \div (2a + b)$.
13. $(x^6 - 729y^6) \div (x - 3y)$.
14. $(x^6 - 729y^6) \div (x + 3y)$.
15. $(81a^4 - 16c^4) \div (3a - 2c)$.
16. $(81a^4 - 16c^4) \div (3a + 2c)$.
17. $(256a^4 - 10,000) \div (4a - 10)$.
18. $(256a^4 - 10,000) \div (4a + 10)$.
19. $(625x^4 - 1) \div (5x - 1)$.

CASE IV.

It may be easily verified that :

105. *The sum of two equal even powers of two numbers is not divisible by either the sum or the difference of the numbers.*

But when the exponent of each of the two equal powers is composed of an *odd* and an *even* factor, the sum of the given powers is divisible by the sum of the powers expressed by the *even* factor.

Thus, $x^6 + y^6$ is not divisible by $x + y$ or by $x - y$, but is divisible by $x^2 + y^2$.

The quotient may be found as in Case II.

EXERCISE 27.

Write by inspection the results in the following examples :

1. $(x^6 + y^6) \div (x^2 + y^2)$.
2. $(a^6 + 1) \div (a^2 + 1)$.
3. $(a^{10} + y^{10}) \div (a^2 + y^2)$.
4. $(b^{10} + 1) \div (b^2 + 1)$.
5. $(a^{12} + b^{12}) \div (a^4 + b^4)$.
6. $(x^{12} + 1) \div (x^4 + 1)$.
7. $(64x^6 + y^6) \div (4x^2 + y^2)$.
8. $(64 + a^6) \div (4 + a^2)$.
9. $(729a^6 + b^6) \div (9a^2 + b^2)$.
10. $(729c^6 + 1) \div (9c^2 + 1)$.

NOTE. The introduction of negative numbers requires an extension of the meaning of some terms common to arithmetic and algebra. But every such extension of meaning must be consistent with the sense previously attached to the term and with general laws already established.

Addition in algebra does not necessarily imply *augmentation*, as it does in arithmetic. Thus, $7 + (-5) = 2$. The word **sum**, however, is used to denote the result.

Such a result is called the **algebraic sum**, when it is necessary to distinguish it from the *arithmetical sum*, which would be obtained by adding the *absolute values* of the numbers.

The general definition of Addition is, the operation of uniting two or more numbers in a *single expression* written in its simplest form.

The general definition of Subtraction is, the operation of finding from two given numbers, called *minuend* and *subtrahend*, a third number, called *difference*, which added to the subtrahend will give the minuend.

The general definition of Multiplication is, the operation of finding from two given numbers, called *multiplicand* and *multiplier*, a third number, called *product*, which may be formed from the multiplicand as the multiplier is formed from unity.

The general definition of Division is, the operation of finding the *other factor* when the *product* of two factors and *one factor* are given.

CHAPTER V.

SIMPLE EQUATIONS.

106. An **equation** is a statement that two expressions are equal. Thus, $4x - 12 = 8$.

107. Every equation consists of two parts, called the first and second *sides*, or *members*, of the equation.

108. An **identical equation** is one in which the two sides are equal, whatever numbers the letters stand for. Thus, $(x + b)(x - b) = x^2 - b^2$.

109. An **equation of condition** is one which is true only when the letters stand for *particular* values. Thus, $x + 5 = 8$ is true only when $x = 3$.

110. A letter to which a particular value must be given in order that the statement contained in an equation may be true is called an *unknown number*.

111. The *value* of the unknown number is the **number** which substituted for it will *satisfy* the equation, and is called a *root* of the equation.

112. To **solve** an equation is to find the unknown number.

113. A **simple equation** is one which contains only the *first* power of the symbol for the unknown number, and is also called an equation of the *first degree*.

114. *If equal changes be made in both sides of an equation, the results will be equal.* § 43.

(1) To find the value of x in $x + b = a$.

$$\begin{array}{rcl} & x + b = a ; & \\ \text{Subtract } b \text{ from each side,} & x + b - b = a - b ; & \\ \text{Cancel } +b - b, & x = a - b. & \end{array}$$

(2) To find the value of x in $x - b = a$.

$$\begin{array}{rcl} & x - b = a ; & \\ \text{Subtract } -b \text{ from each side,} & x - b + b = a + b ; & \\ \text{Cancel } -b + b, & x = a + b. & \end{array}$$

The result in each case is the same as if b were transposed to the other side of the equation with its sign changed. Therefore,

115. *Any term may be transposed from one side of an equation to the other provided its sign be changed.*

For, in this transposition, the same number is subtracted from each side of the equation.

116. The signs of all the terms on each side of an equation may be changed ; for this is in effect transposing every term.

117. When the symbols for the known and unknown numbers of an equation are connected by the sign $+$ or $-$, they may be separated by transposing the symbols for the known numbers to one side and the symbols for the unknown to the other.

118. Hence, to solve an equation with one unknown number,

Transpose all the terms involving the unknown number to the left side, and all the other terms to the right side ; combine

the like terms, and divide both sides by the coefficient of the symbol for the unknown number.

119. To verify the result, substitute the value of the symbol for the unknown number in the original equation.

EXERCISE 28.

Find the value of x in

1. $5x - 1 = 19.$

8. $16x - 11 = 7x + 70.$

2. $3x + 6 = 12.$

9. $24x - 49 = 19x - 14.$

3. $24x = 7x + 34.$

10. $3x + 23 = 78 - 2x.$

4. $8x - 29 = 26 - 3x.$

11. $26 - 8x = 80 - 14x.$

5. $12 - 5x = 19 - 12x.$

12. $13 - 3x = 5x - 3.$

6. $3x + 6 - 2x = 7x.$

13. $3x - 22 = 7x + 6.$

7. $5x + 50 = 4x + 56.$

14. $8 + 4x = 12x - 16.$

15. $5x - (3x - 7) = 4x - (6x - 35).$

16. $6x - 2(9 - 4x) + 3(5x - 7) = 10x - (4 + 16x + 35).$

17. $9x - 3(5x - 6) + 30 = 0.$

18. $x - 7(4x - 11) = 14(x - 5) - 19(8 - x) - 61.$

19. $(x + 7)(x - 3) = (x - 5)(x - 15).$

20. $(x - 8)(x + 12) = (x + 1)(x - 6).$

21. $(x - 2)(7 - x) + (x - 5)(x + 3) - 2(x - 1) + 12 = 0.$

22. $(2x - 7)(x + 5) = (9 - 2x)(4 - x) + 229.$

23. $14 - x - 5(x - 3)(x + 2) + (5 - x)(4 - 5x) = 45x - 76.$

24. $(x + 5)^2 - (4 - x)^2 = 21x.$

25. $5(x - 2)^2 + 7(x - 3)^2 = (3x - 7)(4x - 19) + 42.$

EXERCISE 29.

PROBLEMS.

1. Find a number such that when 12 is added to its double the sum shall be 28.

Let x = the number.
 Then $2x$ = its double,
 and $2x + 12$ = double the number increased by 12.
 But 28 = double the number increased by 12.
 $\therefore 2x + 12 = 28,$
 $2x = 28 - 12.$
 $2x = 16,$
 $x = 8.$

2. A farmer had two flocks of sheep, each containing the same number. He sold 21 sheep from one flock and 70 from the other, and then found that he had left in one flock twice as many as in the other. How many had he in each?

Let x = the number of sheep in each flock.
 Then $x - 21$ = the number of sheep left in one flock,
 and $x - 70$ = the number of sheep left in the other.
 $\therefore x - 21 = 2(x - 70),$
 $x - 21 = 2x - 140.$
 $x - 2x = -140 + 21,$
 $-x = -119,$
 $x = 119.$

3. A and B had equal sums of money; B gave A \$5, and then 3 times A's money was equal to 11 times B's money. What had each at first?

Let x = the number of dollars each had.
 Then $x + 5$ = the number of dollars A had after receiving \$5 from B,
 and $x - 5$ = the number of dollars B had after giving A \$5.

$$\begin{aligned}
 \therefore 3(x+5) &= 11(x-5); \\
 3x+15 &= 11x-55; \\
 3x-11x &= -55-15; \\
 -8x &= -70; \\
 x &= 8\frac{1}{2}.
 \end{aligned}$$

Therefore, each had \$8.75.

4. Find a number whose treble exceeds 50 by as much as its double falls short of 40.

$$\begin{aligned}
 \text{Let } x &= \text{the number.} \\
 \text{Then } 3x &= \text{its treble,} \\
 \text{and } 3x-50 &= \text{the excess of its treble over 50;} \\
 \text{also, } 40-2x &= \text{the number its double lacks of 40.} \\
 \therefore 3x-50 &= 40-2x; \\
 3x+2x &= 40+50; \\
 5x &= 90; \\
 x &= 18.
 \end{aligned}$$

5. What two numbers are those whose difference is 14, and whose sum is 48?

$$\begin{aligned}
 \text{Let } x &= \text{the larger number.} \\
 \text{Then } 48-x &= \text{the smaller number,} \\
 \text{and } x-(48-x) &= \text{the difference of the numbers.} \\
 \text{But } 14 &= \text{the difference of the numbers.} \\
 \therefore x-(48-x) &= 14; \\
 x-48+x &= 14; \\
 2x &= 62; \\
 x &= 31.
 \end{aligned}$$

Therefore, the two numbers are 31 and 17.

6. To the double of a certain number I add 14, and obtain as a result 154. What is the number?
7. To four times a certain number I add 16, and obtain as a result 188. What is the number?
8. By adding 46 to a certain number, I obtain as a result a number three times as large as the original number. Find the original number.

9. One number is three times as large as another. If I take the smaller from 16 and the greater from 30, the remainders are equal. What are the numbers?
10. Divide the number 92 into four parts, such that the first exceeds the second by 10, the third by 18, and the fourth by 24.
11. The sum of two numbers is 20; and if three times the smaller number is added to five times the greater, the sum is 84. What are the numbers?
12. The joint ages of a father and son are 80 years. If the age of the son were doubled, he would be 10 years older than his father. What is the age of each?
13. A man has 6 sons, each 4 years older than the next younger. The eldest is three times as old as the youngest. What is the age of each?
14. Add \$24 to a certain sum and the amount will be as much above \$80 as the sum is below \$80. What is the sum?
15. Thirty yards of cloth and 40 yards of silk together cost \$330; and the silk cost twice as much a yard as the cloth. How much did each cost a yard?
16. Find the number whose double increased by 24 exceeds 80 by as much as the number itself is less than 100.
17. The sum of \$500 is divided among A, B, C, and D. A and B have together \$280, A and C \$260, and A and D \$220. How much does each receive?
18. In a company of 266 persons composed of men, women, and children, there are twice as many men as women, and twice as many women as children. How many are there of each?

19. Find two numbers differing by 8, such that four times the less may exceed twice the greater by 10.
20. A is 58 years older than B, and A's age is as much above 60 as B's age is below 50. Find the age of each.
21. A man leaves his property, amounting to \$7500, to be divided among his wife, his two sons, and three daughters, as follows: a son is to have twice as much as a daughter, and the wife \$500 more than all the children together. How much is the share of each?
22. A vessel containing some water was filled by pouring in 42 gallons, and there was then in the vessel seven times as much as at first. How much did the vessel hold?
23. A has \$72 and B has \$52. B gives A a certain sum; then A has three times as much as B. How much did A receive from B?
24. Divide 90 into two such parts that four times one part may be equal to five times the other.
25. Divide 60 into two such parts that one part exceeds the other by 24.
26. Divide 84 into two such parts that one part may be less than the other by 36.

NOTE I. When we have to compare the ages of two persons at a given time, and also a number of years after or before the given time, we must remember that *both* persons will be so many years older or younger.

Thus, if x represent A's age, and $2x$ B's age, at the present time, A's age five years ago will be represented by $x - 5$; and B's by $2x - 5$. A's age five years hence will be represented by $x + 5$; and B's age by $2x + 5$.

27. A is twice as old as B, and 22 years ago he was three times as old as B. What is A's age?
28. A father is 30 and his son 6 years old. In how many years will the father be just twice as old as the son?
29. A is twice as old as B, and 20 years ago he was three times as old. What is B's age?
30. A is three times as old as B, and 19 years hence he will be only twice as old as B. What is the age of each?
31. A man has three nephews; his age is 50, and the joint ages of the nephews is 42. How long will it be before the joint ages of the nephews will be equal to that of the uncle?

NOTE II. In problems involving quantities of the same kind expressed in different units, we must be careful to reduce all the quantities to the *same unit*.

Thus, if x denotes a number of inches, all the quantities of the same kind involved in the problem must be reduced to inches.

32. A sum of money consists of dollars and twenty-five-cent pieces, and amounts to \$20. The number of coins is 50. How many are there of each sort?
33. A person bought 30 pounds of sugar of two different kinds, and paid for the whole \$2.94. The better kind cost 10 cents a pound and the poorer kind 7 cents a pound. How many pounds were there of each kind?
34. A workman was hired for 40 days, at \$1 for every day he worked, but with the condition that for every day he did not work he was to pay 45 cents for his board. At the end of the time he received \$22.60. How many days did he work?

35. A wine merchant has two kinds of wine; one worth 50 cents a quart, and the other 75 cents a quart. From these he wishes to make a mixture of 100 gallons, worth \$2.40 a gallon. How many gallons must he take of each kind?
36. A gentleman gave some children 10 cents each, and had a dollar left. He found that he would have required one dollar more to enable him to give them 15 cents each. How many children were there?
37. Two casks contain equal quantities of vinegar; from the first cask 34 quarts are drawn, from the second, 20 gallons; the quantity remaining in one vessel is now twice that in the other. How much did each cask contain at first?
38. A gentleman hired a man for 12 months, at the wages of \$90 and a suit of clothes. At the end of 7 months the man quits his service and receives \$33.75 and the suit of clothes. What was the price of the suit of clothes?
39. A man has three times as many quarters as half-dollars, four times as many dimes as quarters, and twice as many half-dimes as dimes. The whole sum is \$7.30. How many coins has he in all?
40. A person paid a bill of \$15.25 with quarters and half-dollars, and gave 51 pieces of money all together. How many of each kind were there?
41. A bill of 100 pounds was paid with guineas (21 shillings) and half-crowns ($2\frac{1}{2}$ shillings), and 48 more half-crowns than guineas were used. How many of each were paid?

CHAPTER VI.

FACTORS.

120. An expression is *rational* if none of its terms contains square or other roots. It is often important to determine the *rational* and *integral factors* of a given expression.

121. CASE I. The simplest case is that in which all the terms of an expression have one common factor. Thus,

$$(1) \ x^2 + xy = x(x + y).$$

$$(2) \ 6a^3 + 4a^2 + 8a = 2a(3a^2 + 2a + 4).$$

$$(3) \ 18a^3b - 27a^2b^2 + 36ab = 9ab(2a^2 - 3ab + 4).$$

EXERCISE 30.

Resolve into factors :

1. $5a^2 - 15a.$

4. $4x^3y - 12x^2y^2 + 8xy^3.$

2. $6a^3 + 18a^2 - 12a.$

5. $y^4 - ay^3 + by^2 + cy.$

3. $49x^2 - 21x + 14.$

6. $6a^5b^3 - 21a^4b^2 + 27a^3b^4.$

7. $54x^2y^2 + 108x^4y^2 - 243x^6y^2.$

8. $45x^7y^{10} - 90x^5y^7 - 360x^4y^3.$

9. $70a^3y^4 - 140a^2y^3 + 210ay^2.$

10. $32a^3b^6 + 96a^6b^3 - 128a^6b^9.$

122. CASE II Frequently the terms of an expression can be so arranged as to show a common factor. Thus,

$$\begin{aligned}(1) \quad x^2 + ax + bx + ab &= (x^2 + ax) + (bx + ab), \\ &= x(x + a) + b(x + a), \\ &= (x + b)(x + a).\end{aligned}$$

$$\begin{aligned}(2) \quad ac - ad - bc + bd &= (ac - ad) - (bc - bd), \\ &= a(c - d) - b(c - d), \\ &= (a - b)(c - d).\end{aligned}$$

EXERCISE 31.

Resolve into factors :

- | | |
|--------------------------------|------------------------------------|
| 1. $x^2 - ax - bx + ab$. | 6. $abx - aby + pqx - pqy$. |
| 2. $ab + ay - by - y^2$. | 7. $cdx^2 + adxy - bcxy - aby^2$. |
| 3. $bc + bx - cx - x^2$. | 8. $abcy - b^2dy - acdx + bd^2x$. |
| 4. $mx + mn + ax + an$. | 9. $ax - ay - bx + by$. |
| 5. $cdx^2 - cxy + dxy - y^2$. | 10. $cdx^2 - cyz + dyz - y^2$. |

123. The square root of a number is one of the *two equal* factors of that number. Thus, the square root of 25 is 5 ; for, $25 = 5 \times 5$.

The square root of a^4 is a^2 ; for, $a^4 = a^2 \times a^2$.

The square root of $a^2b^2c^2$ is abc ; for, $a^2b^2c^2 = abc \times abc$.

In general, the square root of a power of an expression is found by taking the square root of the numerical coefficient and writing each letter with an exponent equal to one-half the exponent of the letter in the given power.

The square root of a product may be found by taking the square root of each factor, and finding the product of the roots.

The square root of a positive number may be either positive or negative; for,

$$a^2 = a \times a,$$

or,

$$a^2 = -a \times -a;$$

but throughout this chapter only the positive value of the square root will be taken.

124. CASE III. From § 73 it is seen that a trinomial is often the product of two binomials. Conversely, a trinomial may, in certain cases, be resolved into two binomial factors. Thus,

To find the factors of

$$x^2 + 7x + 12.$$

The first term of each binomial factor will obviously be x .

The second terms of the two binomial factors must be two numbers

whose *product* is 12,

and

whose *sum* is 7.

The only two numbers whose product is 12 and whose sum is 7 are 4 and 3.

$$\therefore x^2 + 7x + 12 = (x + 4)(x + 3).$$

Again, to find the factors of $x^2 + 5xy + 6y^2$.

The first term of each binomial factor will obviously be x .

The second terms of the two binomial factors must be two numbers

whose *product* is $6y^2$,

and

whose *sum* is $5y$.

The only two numbers whose product is $6y^2$ and whose sum is $5y$ are $3y$ and $2y$.

$$\therefore x^2 + 5xy + 6y^2 = (x + 3y)(x + 2y).$$

EXERCISE 32.

Find the factors of :

- | | |
|--------------------------------|--|
| 1. $x^2 + 11x + 24$. | 11. $x^2 + 13ax + 36a^2$. |
| 2. $x^2 + 11x + 30$. | 12. $y^2 + 19py + 48p^2$. |
| 3. $y^2 + 17y + 60$. | 13. $z^2 + 29qz + 100q^2$. |
| 4. $z^2 + 13z + 12$. | 14. $a^4 + 5a^2 + 6$. |
| 5. $x^2 + 21x + 110$. | 15. $z^6 + 4z^3 + 3$. |
| 6. $y^2 + 35y + 300$. | 16. $a^2b^2 + 18ab + 32$. |
| 7. $b^2 + 23b + 102$. | 17. $x^3y^4 + 7x^4y^2 + 12$. |
| 8. $x^2 + 3x + 2$. | 18. $z^{10} + 10z^5 + 16$. |
| 9. $x^2 + 7x + 6$. | 19. $a^2 + 9ab + 20b^2$. |
| 10. $a^2 + 9ab + 8b^2$. | 20. $x^6 + 9x^3 + 20$. |
| 21. $a^2x^2 + 14abx + 33b^2$. | 24. $b^2c^2 + 18abc + 65a^2$. |
| 22. $a^2c^2 + 7acx + 10x^2$. | 25. $r^2s^2 + 23rsz + 90z^2$. |
| 23. $x^2y^2z^2 + 19xyz + 48$. | 26. $m^4n^4 + 20m^2n^2pq + 51p^2q^2$. |

125. CASE IV. To find the factors of

$$x^2 - 9x + 20.$$

The second terms of the two binomial factors must be two numbers

whose *product* is 20,

and

whose *sum* is -9 .

The only two numbers whose product is 20 and whose sum is -9 are -5 and -4 .

$$\therefore x^2 - 9x + 20 = (x - 5)(x - 4).$$

EXERCISE 33.

Resolve into factors :

- | | |
|-------------------------------|------------------------------------|
| 1. $x^2 - 7x + 10.$ | 13. $a^2b^2c^2 - 13abc + 22.$ |
| 2. $x^2 - 29x + 190.$ | 14. $x^2 - 15x + 50.$ |
| 3. $a^2 - 23a + 132.$ | 15. $x^2 - 20x + 100.$ |
| 4. $b^2 - 30b + 200.$ | 16. $a^2x^2 - 21ax + 54.$ |
| 5. $z^2 - 43z + 460.$ | 17. $a^2x^2 - 16abx + 39b^2.$ |
| 6. $x^2 - 7x + 6.$ | 18. $a^2c^2 - 24acz + 143z^2.$ |
| 7. $x^4 - 4a^2x^2 + 3a^4.$ | 19. $x^2 - 20x + 91.$ |
| 8. $x^2 - 8x + 12.$ | 20. $x^2 - 23x + 120.$ |
| 9. $z^2 - 57z + 56.$ | 21. $z^2 - 53z + 360.$ |
| 10. $y^6 - 7y^3 + 12.$ | 22. $x^2 - (a + c)x + ac.$ |
| 11. $x^2y^2 - 27xy + 26.$ | 23. $y^2z^2 - 28abyz + 187a^2b^2.$ |
| 12. $a^4b^6 - 11a^2b^3 + 30.$ | 24. $c^2d^2 - 30abcd + 221a^2b^2.$ |

126. CASE V. To find the factors of

$$x^2 + 2x - 3.$$

The second terms of the two binomial factors must be two numbers

whose *product* is -3 ,
and whose *sum* is $+2$.

The only two numbers whose product is -3 and whose sum is $+2$ are $+3$ and -1 .

$$\therefore x^2 + 2x - 3 = (x + 3)(x - 1).$$

EXERCISE 34.

Resolve into factors :

- | | |
|------------------------|-------------------------------|
| 1. $x^2 + 6x - 7$. | 8. $a^2 + 25a - 150$. |
| 2. $x^2 + 5x - 84$. | 9. $b^8 + 3b^4 - 4$. |
| 3. $y^2 + 7y - 60$. | 10. $b^2c^2 + 3bc - 154$. |
| 4. $y^2 + 12y - 45$. | 11. $c^{10} + 15c^5 - 100$. |
| 5. $z^2 + 11z - 12$. | 12. $c^3 + 17c - 390$. |
| 6. $z^2 + 13z - 140$. | 13. $a^2 + a - 132$. |
| 7. $a^2 + 13a - 300$. | 14. $x^2y^2z^2 + 9xyz - 22$. |

127. CASE VI. To find the factors of

$$x^2 - 5x - 66.$$

The second terms of the two binomial factors must be two numbers

whose *product* is -66 ,

and

whose *sum* is -5 .

The only two numbers whose product is -66 and whose sum is -5 are -11 and $+6$.

$$\therefore x^2 - 5x - 66 = (x - 11)(x + 6).$$

EXERCISE 35.

Resolve into factors :

- | | |
|-----------------------|---------------------------|
| 1. $x^2 - 3x - 28$. | 6. $a^2 - 15a - 100$. |
| 2. $y^2 - 7y - 18$. | 7. $c^{10} - 9c^5 - 10$. |
| 3. $x^2 - 9x - 36$. | 8. $x^2 - 8x - 20$. |
| 4. $z^2 - 11z - 60$. | 9. $y^2 - 5ay - 50a^2$. |
| 5. $z^2 - 13z - 14$. | 10. $a^2b^2 - 3ab - 4$. |

11. $a^2x^3 - 3ax - 54.$

14. $y^2z^4 - 5y^4z^2 - 84.$

12. $c^2d^2 - 24cd - 180.$

15. $a^2b^2 - 16ab - 36.$

13. $a^6c^2 - a^3c - 2.$

16. $x^2 - (a - b)x - ab.$

We now proceed to the consideration of trinomials which are perfect squares. These are only particular forms of Cases III. and IV., but from their importance demand special attention.

128. CASE VII. To find the factors of

$$x^2 + 18x + 81.$$

The second terms of the two binomial factors must be two numbers

whose *product* is 81,

and

whose *sum* is 18.

The only two numbers whose product is 81 and whose sum is 18 are 9 and 9.

$$\therefore x^2 + 18x + 81 = (x + 9)(x + 9) = (x + 9)^2.$$

EXERCISE 36.

Resolve into factors :

1. $x^2 + 12x + 36.$

8. $y^4 + 16y^2z^2 + 64z^4.$

2. $x^2 + 28x + 196.$

9. $y^5 + 24y^3 + 144.$

3. $x^2 + 34x + 289.$

10. $x^3z^2 + 162xz + 6561.$

4. $z^2 + 2z + 1.$

11. $4a^3 + 12ab^2 + 9b^4.$

5. $y^2 + 200y + 10,000.$

12. $9x^2y^4 + 30xy^2z + 25z^2.$

6. $z^4 + 14z^2 + 49.$

13. $9x^2 + 12xy + 4y^2.$

7. $x^2 + 36xy + 324y^2.$

14. $4a^4x^2 + 20a^2x^3y + 25x^4y^2.$

129. CASE VIII. To find the factors of

$$x^2 - 18x + 81.$$

The second terms of the two binomials must be two numbers

whose *product* is 81,

and

whose *sum* is -18 .

The only two numbers whose product is 81 and whose sum is -18 are -9 and -9 .

$$\therefore x^2 - 18x + 81 = (x - 9)(x - 9) = (x - 9)^2.$$

EXERCISE 37.

Resolve into factors :

- | | |
|---|--|
| 1. $a^2 - 8a + 16$. | 10. $4x^4y^2 - 20x^2y^2z + 25y^4z^2$. |
| 2. $a^2 - 30a + 225$. | 11. $16x^2y^4 - 8xy^2z^2 + y^2z^4$. |
| 3. $x^2 - 38x + 361$. | 12. $9a^2b^2c^2 - 6ab^2c^2d + b^2c^2d^2$. |
| 4. $x^2 - 40x + 400$. | 13. $16x^6 - 8x^4y^2 + x^2y^4$. |
| 5. $y^2 - 100y + 2500$. | 14. $a^6x^4 - 2a^3bx^2y^4 + b^2y^8$. |
| 6. $y^4 - 20y^2 + 100$. | 15. $36x^2y^2 - 60xy^3 + 25y^4$. |
| 7. $y^2 - 50yz + 625z^2$. | 16. $1 - 6ab^3 + 9a^2b^6$. |
| 8. $x^4 - 32x^2y^2 + 256y^4$. | 17. $9m^2n^2 - 24mn + 16$. |
| 9. $z^5 - 34z^2 + 289$. | 18. $4b^2x^2 - 12bx^2y + 9x^2y^2$. |
| 19. $49a^2 - 112ab + 64b^2$. | |
| 20. $64x^4y^6 - 160x^4y^3z + 100x^4z^2$. | |
| 21. $49a^2b^2c^2 - 28abcx + 4x^2$. | |
| 22. $121x^4 - 286x^2y + 169y^2$. | |
| 23. $289x^2y^2z^2 - 102xy^2z^2d + 9y^2z^2d^2$. | |
| 24. $361x^2y^2z^2 - 76abcxyz + 4a^2b^2c^2$. | |

130. CASE IX. An expression in the form of two squares, with the negative sign between them, is the product of two factors which may be determined as follows:

Take the square root of the first term, and the square root of the second term.

The *sum* of these roots will form the first factor;

The *difference* of these roots will form the second factor.
Thus:

$$(1) \ a^2 - b^2 = (a + b)(a - b).$$

$$(2) \ a^2 - (b - c)^2 = \{a + (b - c)\}\{a - (b - c)\}, \\ = \{a + b - c\}\{a - b + c\}.$$

$$(3) \ (a - b)^2 - (c - d)^2 = \{(a - b) + (c - d)\}\{(a - b) - (c - d)\}, \\ = \{a - b + c - d\}\{a - b - c + d\}.$$

131. The terms of an expression may often be arranged so as to form two squares with the negative sign between them, and the expression can then be resolved into factors.
Thus:

$$\begin{aligned} & a^2 + b^2 - c^2 - d^2 + 2ab + 2cd, \\ & = a^2 + 2ab + b^2 - c^2 + 2cd - d^2, \\ & = (a^2 + 2ab + b^2) - (c^2 - 2cd + d^2), \\ & = (a + b)^2 - (c - d)^2, \\ & = \{(a + b) + (c - d)\}\{(a + b) - (c - d)\}, \\ & = \{a + b + c - d\}\{a + b - c + d\}. \end{aligned}$$

132. An expression may often be resolved into three or four factors. Thus:

$$\begin{aligned} (1) \ x^{16} - y^{16} &= (x^8 + y^8)(x^8 - y^8) \\ &= (x^8 + y^8)(x^4 + y^4)(x^4 - y^4) \\ &= (x^8 + y^8)(x^4 + y^4)(x^2 + y^2)(x^2 - y^2) \\ &= (x^8 + y^8)(x^4 + y^4)(x^2 + y^2)(x + y)(x - y). \end{aligned}$$

$$\begin{aligned}
(2) \quad & 4(ab+cd)^2 - (a^2+b^2-c^2-d^2)^2, \\
& = \{2(ab+cd) + (a^2+b^2-c^2-d^2)\} \\
& \quad \{2(ab+cd) - (a^2+b^2-c^2-d^2)\}, \\
& = \{2ab+2cd+a^2+b^2-c^2-d^2\} \\
& \quad \{2ab+2cd-a^2-b^2+c^2+d^2\}, \\
& = \{(a^2+2ab+b^2) - (c^2-2cd+d^2)\} \\
& \quad \{(c^2+2cd+d^2) - (a^2-2ab+b^2)\}, \\
& = \{(a+b)^2 - (c-d)^2\} \{(c+d)^2 - (a-b)^2\}, \\
& = \{a+b+(c-d)\} \{a+b-(c-d)\} \\
& \quad \{c+d+(a-b)\} \{c+d-(a-b)\}, \\
& = \{a+b+c-d\} \{a+b-c+d\} \\
& \quad \{c+d+a-b\} \{c+d-a+b\}.
\end{aligned}$$

EXERCISE 38.

Resolve into factors :

- | | |
|------------------------------|---|
| 1. $a^2 - b^2$. | 14. $(a+b)^2 - (c+d)^2$. |
| 2. $a^2 - 16$. | 15. $(x+y)^2 - (x-y)^2$. |
| 3. $4a^2 - 25$. | 16. $2ab - a^2 - b^2 + 1$. |
| 4. $a^4 - b^4$. | 17. $x^2 - 2yz - y^2 - z^2$. |
| 5. $a^4 - 1$. | 18. $x^2 - 2xy + y^2 - z^2$. |
| 6. $a^8 - b^8$. | 19. $a^2 + 12bc - 4b^2 - 9c^2$. |
| 7. $a^8 - 1$. | 20. $a^2 - 2ay + y^2 - x^2 - 2xz - z^2$. |
| 8. $36x^2 - 49y^2$. | 21. $2xy - x^2 - y^2 + z^2$. |
| 9. $100x^2y^2 - 121a^2b^2$. | 22. $x^2 + y^2 - z^2 - d^2 - 2xy - 2dz$. |
| 10. $1 - 49x^2$. | 23. $x^2 - y^2 + z^2 - a^2 - 2xz + 2ay$. |
| 11. $a^4 - 25b^2$. | 24. $2ab + a^2 + b^2 - c^2$. |
| 12. $(a-b)^2 - c^2$. | 25. $2xy - x^2 - y^2 + a^2 + b^2 - 2ab$. |
| 13. $x^2 - (a-b)^2$. | 26. $(ax+by)^2 - 1$. |

27. $1 - x^2 - y^2 + 2xy$. 31. $(x+1)^2 - (y-1)^2$.
 28. $(5a-2)^2 - (a-4)^2$. 32. $d^2 - x^2 + 4xy - 4y^2$.
 29. $a^2 - 2ab + b^2 - x^2$. 33. $a^2 - b^2 - 2bc - c^2$.
 30. $(x+1)^2 - (y+1)^2$. 34. $4x^4 - 9x^2 + 6x - 1$.

133. CASE X.

Since
$$\frac{x^3 - y^3}{x - y} = x^2 + xy + y^2,$$

and
$$\frac{x^5 - y^5}{x - y} = x^4 + x^3y + x^2y^2 + xy^3 + y^4,$$

and so on, it follows that the difference between two equal odd powers of two numbers is divisible by the difference between the numbers.

EXERCISE 39.

Resolve into factors:

- | | |
|------------------|---------------------------|
| 1. $a^3 - b^3$. | 6. $8x^3 - 27y^3$. |
| 2. $x^3 - 8$. | 7. $64y^3 - 1000z^3$. |
| 3. $x^3 - 343$. | 8. $729x^3 - 512y^3$. |
| 4. $y^3 - 125$. | 9. $27a^3 - 1728$. |
| 5. $y^3 - 216$. | 10. $1000a^3 - 1331b^3$. |

134. CASE XI.

Since
$$\frac{x^3 + a^3}{x + a} = x^2 - ax + a^2,$$

and
$$\frac{x^5 + y^5}{x + y} = x^4 - x^3y + x^2y^2 - xy^3 + y^4,$$

and so on, it follows that the sum of two equal odd powers of two numbers is divisible by the sum of the numbers.

EXERCISE 40.

Resolve into factors :

- | | |
|-----------------------|-------------------------|
| 1. $x^3 + y^3$. | 6. $216a^3 + 512c^3$. |
| 2. $x^3 + 8$. | 7. $729x^3 + 1728y^3$. |
| 3. $x^3 + 216$. | 8. $x^3 + y^3$. |
| 4. $y^3 + 64z^3$. | 9. $x^3 + y^3$. |
| 5. $64b^3 + 125c^3$. | 10. $32b^3 + 243c^3$. |

135. CASE XII. The sum of *any two powers* of two numbers, whose exponents contain the *same odd* factor, is divisible by the sum of the powers obtained by dividing the exponents of the given powers by this *odd factor*.

Thus,

$$\frac{x^6 + y^6}{x^3 + y^3} = x^3 - x^2y^3 + y^3.$$

$$\frac{x^6 + y^6}{x^3 + y^3} = x^3 - x^2y^3 + y^3.$$

In like manner, $x^{10} + 32y^5$, which is equal to $x^{10} + (2y)^5$, is divisible by $x^2 + 2y$; but $x^4 + y^4$, whose exponents do not contain *an odd* factor, and $x^6 + y^{10}$, whose exponents do not contain the *same odd* factor, cannot be resolved into rational factors.

EXERCISE 41.

Resolve into factors :

- | | | | |
|------------------------|------------------------|-------------------|--------------------|
| 1. $a^6 + b^6$. | 3. $x^{12} + y^{12}$. | 5. $x^6 + 1$. | 7. $64a^6 + x^6$. |
| 2. $a^{10} + b^{10}$. | 4. $b^6 + 64c^6$. | 6. $a^{12} + 1$. | 8. $729 + c^6$. |

136. CASE XIII. For a trinomial to be a perfect square, *the middle term must be twice the product of the square roots of the first and last terms.*

The expression $x^4 + x^2y^2 + y^4$ will become a perfect square if x^2y^2 be added to the middle term. And if the subtraction of x^2y^2 from the expression thus obtained be indicated, the result will be the difference of two squares. Thus :

$$\begin{aligned} x^4 + x^2y^2 + y^4 &= (x^4 + 2x^2y^2 + y^4) - x^2y^2, \\ &= (x^2 + y^2)^2 - x^2y^2, \\ &= (x^2 + y^2 + xy)(x^2 + y^2 - xy), \\ \text{or, } &(x^2 + xy + y^2)(x^2 - xy + y^2). \end{aligned}$$

EXERCISE 42.

Resolve into factors :

- | | |
|---------------------------------|-----------------------------------|
| 1. $a^4 + a^2b^2 + b^4$. | 8. $49m^4 + 110m^2n^2 + 81n^4$. |
| 2. $9x^4 + 3x^2y^2 + 4y^4$. | 9. $9a^4 + 21a^2c^2 + 25c^4$. |
| 3. $16x^4 - 17x^2y^2 + y^4$. | 10. $49a^4 - 15a^2b^2 + 121b^4$. |
| 4. $81a^4 + 23a^2b^2 + 16b^4$. | 11. $64x^4 + 128x^2y^2 + 81y^4$. |
| 5. $81a^4 - 28a^2b^2 + 16b^4$. | 12. $4x^4 - 37x^2y^2 + 9y^4$. |
| 6. $9x^4 + 38x^2y^2 + 49y^4$. | 13. $25x^4 - 41x^2y^2 + 16y^4$. |
| 7. $25a^4 - 9a^2b^2 + 16b^4$. | 14. $81x^4 - 34x^2y^2 + y^4$. |

* If, in Example 12, $9y^4 = (-3y^2)^2$, then $25x^2y^2$ should be added to $4x^4 - 37x^2y^2 + 9y^4$, in order to make the expression a perfect square. That is, we should have :

$$\begin{aligned} &(4x^4 - 12x^2y^2 + 9y^4) - 25x^2y^2, \\ &= (2x^2 - 3y^2)^2 - 25x^2y^2, \\ &= (2x^2 - 3y^2 + 5xy)(2x^2 - 3y^2 - 5xy), \\ \text{or, } &(2x^2 + 5xy - 3y^2)(2x^2 - 5xy - 3y^2). \end{aligned}$$

137. CASE XIV. To find the factors of

$$6x^2 + x - 12.$$

It is evident that the first terms of the two factors might be $6x$ and x , or $2x$ and $3x$, since the product of either of these pairs is $6x^2$.

Likewise, the last terms of the two factors might be 12 and 1, 6 and 2, or 4 and 3 (if we disregard the signs).

From these it is necessary to select such as will produce the middle term of the trinomial. And they are found by trial to be $3x$ and $2x$, and -4 and $+3$.

$$\therefore 6x^2 + x - 12 = (3x - 4)(2x + 3).$$

EXERCISE 43.

Resolve into factors:

- | | |
|-----------------------------|----------------------------|
| 1. $12x^2 - 5x - 2.$ | 13. $6a^2x^2 + ax - 1.$ |
| 2. $12x^2 - 7x + 1.$ | 14. $6b^2 - 7bx - 3x^2.$ |
| 3. $12x^2 - x - 1.$ | 15. $4x^2 + 8x + 3.$ |
| 4. $3x^2 - 2x - 5.$ | 16. $a^2 - ax - 6x^2.$ |
| 5. $3x^2 + 4x - 4.$ | 17. $8a^2 + 14ab - 15b^2.$ |
| 6. $6x^2 + 5x - 4.$ | 18. $6a^2 - 19ac + 10c^2.$ |
| 7. $4x^2 + 13x + 3.$ | 19. $8x^2 + 34xy + 21y^2.$ |
| 8. $4x^2 + 11x - 3.$ | 20. $8x^2 - 22xy - 21y^2.$ |
| 9. $4x^2 - 4x - 3.$ | 21. $6x^2 + 19xy - 7y^2.$ |
| 10. $x^3 - 3ax + 2a^2.$ | 22. $11a^2 - 23ab + 2b^2.$ |
| 11. $12a^4 + a^2x^2 - x^4.$ | 23. $2c^2 - 13cd + 6d^2.$ |
| 12. $2x^2 + 5xy + 2y^2.$ | 24. $6y^2 + 7yz - 3z^2.$ |

138. CASE XV. The factors, if any exist, of a polynomial of more than three terms can often be found by the application of principles already explained. Thus it is seen at a glance that the expression

$$a^3 - 3a^2b + 3ab^2 - b^3$$

fulfills, both in respect to exponents and coefficients, the laws stated in § 83 for writing the power of a binomial; and it is known at once that

$$a^3 - 3a^2b + 3ab^2 - b^3 = (a - b)^3.$$

Again, it is seen that the expression

$$x^3 - 2xy + y^2 + 2xz - 2yz + z^2$$

consists of *three* squares and *three* double products, and from § 79, is the square of a *trinomial* which has for terms x, y, z .

It is also seen from the double product $-2xy$, that x and y have *unlike* signs;

and from the double product $2xz$, that x and z have *like* signs. Hence,

$$x^3 - 2xy + y^2 + 2xz - 2yz + z^2 = (x - y + z)^2.$$

EXERCISE 44.

Resolve into factors:

$$1. a^3 + 3a^2b + 3ab^2 + b^3. \quad 4. x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4.$$

$$2. a^3 + 3a^2 + 3a + 1. \quad 5. x^4 - 4x^3 + 6x^2 - 4x + 1.$$

$$3. a^3 - 3a^2 + 3a - 1. \quad 6. a^4 - 4a^3c + 6a^2c^2 - 4ac^3 + c^4.$$

$$7. x^2 + 2xy + y^2 + 2xz + 2yz + z^2.$$

$$8. x^2 - 2xy + y^2 - 2xz + 2yz + z^2.$$

$$9. a^2 + b^2 + c^2 + 2ab - 2ac - 2bc.$$

139. CASE XVI. Multiply $2x - y + 3$ by $x + 2y - 3$.

$$\begin{array}{r}
 2x - y + 3 \\
 x + 2y - 3 \\
 \hline
 2x^2 - xy \qquad + 3x \\
 \qquad 4xy - 2y^2 \qquad + 6y \\
 \qquad \qquad - 6x + 3y - 9 \\
 \hline
 2x^2 + 3xy - 2y^2 - 3x + 9y - 9
 \end{array}$$

It is to be observed that $2x^2 + 3xy - 2y^2$, of the product, is obtained from $(2x - y) \times (x + 2y)$;

that -9 is obtained from 3×-3 ;

that $-3x$ is the sum of $2x \times -3$ and $x \times 3$;

that $9y$ is the sum of $2y \times 3$ and $-y \times -3$.

From this result may be deduced a method of resolving into its factors a polynomial which is composed of two trinomial factors. Thus:

Find the factors of

$$6x^2 - 7xy - 3y^2 - 9x + 30y - 27.$$

The factors of the first three terms are (by Case XIV.)

$$3x + y \text{ and } 2x - 3y$$

Now -27 must be resolved into two factors such that the sum of the products obtained by multiplying one of these factors by $3x$ and the other by $2x$ shall be $-9x$.

These two factors evidently are -9 and $+3$.

$$\begin{aligned}
 \text{That is, } (6x^2 - 7xy - 3y^2 - 9x + 30y - 27) \\
 = (3x + y - 9)(2x - 3y + 3).
 \end{aligned}$$

140. The following method is often most convenient for separating a polynomial into its factors:

Find the factors of

$$2x^2 - 5xy + 2y^2 + 7xz - 5yz + 3z^2.$$

1. Reject the terms that contain z .
2. Reject the terms that contain y .
3. Reject the terms that contain x .

Factor the expression that remains in each case.

1. $2x^2 - 5xy + 2y^2 = (x - 2y)(2x - y).$

2. $2x^2 + 7xz + 3z^2 = (x + 3z)(2x + z).$

3. $2y^2 - 5yz + 3z^2 = (-2y + 3z)(-y + z).$

Arrange these three pairs of factors in two rows of three factors each, so that any two factors of each row may have a *common term*. Thus :

$$\begin{array}{l} x - 2y, x + 3z, -2y + 3z; \\ 2x - y, 2x + z, -y + z. \end{array}$$

From the first row, select the *terms common to two factors* for one trinomial factor :

$$x - 2y + 3z.$$

From the second row, select the *terms common to two factors* for the other trinomial factor :

$$2x - y + z.$$

$$\begin{aligned} \text{Then, } 2x^2 - 5xy + 2y^2 + 7xz - 5yz + 3z^2 \\ = (x - 2y + 3z)(2x - y + z). \end{aligned}$$

141. When a factor obtained from the first three terms is also a factor of the remaining terms, the expression is easily resolved. Thus :

$$\begin{aligned} x^3 - 3xy + 2y^2 - 3x + 6y, \\ = (x - 2y)(x - y) - 3(x - 2y), \\ = (x - 2y)(x - y - 3). \end{aligned}$$

EXERCISE 45.

Resolve into factors :

1. $2x^2 - 5xy + 2y^2 - 17x + 13y + 21.$

2. $6x^2 - 37xy + 6y^2 - 5x - 5y - 1.$

3. $6x^2 - 5xy - 6y^2 - x - 5y - 1.$

4. $5x^2 - 8xy + 3y^2 + 7x - 5y + 2.$

5. $2x^2 - xy - 3y^2 - 8x + 7y + 6.$

6. $x^2 - 25y^2 - 10x - 20y + 21$.
7. $2x^2 - 5xy + 2y^2 - xz - yz - z^2$.
8. $6x^2 + xy - y^2 - 3xz + 6yz - 9z^2$.
9. $6x^2 - 7xy + y^2 + 35xz - 5yz - 6z^2$.
10. $5x^2 - 8xy + 3y^2 - 3xz + yz - 2z^2$.
11. $2x^2 - xy - 3y^2 - 5yz - 2z^2$.
12. $6x^2 - 13xy + 6y^2 + 12xz - 13yz + 6z^2$.
13. $x^2 - 2xy + y^2 + 5x - 5y$.
14. $2x^2 + 5xy - 3y^2 - 4xz + 2yz$.

EXERCISE 46.

MISCELLANEOUS EXAMPLES.

The following expressions are to be resolved into factors by the principles already explained. The student should first carefully remove all monomial factors from the expressions.

- | | |
|--|---------------------------------|
| 1. $5x^2 - 15x - 20$. | 9. $a^4 + a^2 + 1$. |
| 2. $2x^5 - 16x^4 + 24x^3$. | 10. $x^3 - y^2 - xz + yz$. |
| 3. $3a^2b^2 - 9ab - 12$. | 11. $ab - ac - b^2 + bc$. |
| 4. $a^2 + 2ax + x^2 + 4a + 4x$. | 12. $3x^2 - 3xz - xy + yz$. |
| 5. $a^2 - 2ab + b^2 - c^2$. | 13. $a^2 - x^2 - ab - bx$. |
| 6. $x^2 - 2xy + y^2 - c^2 + 2cd - d^2$. | 14. $a^2 - 2ax + x^2 + a - x$. |
| 7. $4 - x^2 - 2x^3 - x^4$. | 15. $3x^2 - 3y^2 - 2x + 2y$. |
| 8. $a^2 - b^2 - a - b$. | 16. $x^4 + x^3 + x^2 + x$. |
| 17. $a^4x^4 - a^3x^3 - a^2x^2 + 1$. | |
| 18. $3x^3 - 2x^2y - 27xy^2 + 18y^3$. | |

19. $4x^4 - x^3 + 2x - 1$. 28. $4a^3 - 4ab + b^3$.
 20. $x^6 - y^6$. 29. $16x^3 - 80xy + 100y^2$.
 21. $x^6 + y^6$. 30. $36a^2x^3y^2 - 25b^2x^5y^2$.
 22. $729 - x^6$. 31. $9x^3y^4 - 30xy^2z + 25z^3$.
 23. $x^{12}y + y^{12}$. 32. $16x^5 - x$.
 24. $a^4c - c^5$. 33. $x^3 - 2xy - 2xz + y^3 + 2yz + z^3$.
 25. $x^3 + 4x - 21$. 34. $a^2 - ab - 6b^2 - 4a + 12b$.
 26. $3a^2 - 21ab + 30b^2$. 35. $x^3 + 2xy + y^3 - x - y - 6$.
 27. $2x^4 - 4x^3y - 6x^2y^2$. 36. $(a + b)^4 - c^4$.
 37. $x^3 - xy - 6y^2 - 4x + 12y$. 39. $3x^3 - 11xy + 6y^2$.
 38. $1 - x + x^2 - x^3$. 40. $x^2 + 20x + 91$.
 41. $(x - y)(x^2 - z^2) - (x - z)(x^2 - y^2)$.
 42. $x^3 - 5x - 24$. 50. $y^2 - 4y - 117$.
 43. $(x^3 - y^3 - z^3)^2 - 4y^2z^3$. 51. $x^2 + 6x - 135$.
 44. $5x^3y^3 + 5x^2yz - 60xz^3$. 52. $4a^3 - 12ab + 9b^2 - 4c^3$.
 45. $3x^3 - x^3 + 3x - 1$. 53. $(a + 3b)^2 - 9(b - c)^2$.
 46. $x^2 - 2mx + m^2 - n^2$. 54. $9x^3 - 4y^3 + 4yz - z^3$.
 47. $4a^3b^3 - (a^2 + b^2 - c^2)^3$. 55. $6b^2x^3 - 7bx^3 - 3x^4$.
 48. $a^7 + a^5$. 56. $a^3 - b^3 - 3ab(a - b)$.
 49. $1 - 14a^3x + 49a^5x^2$. 57. $x^3 + y^3 + 3xy(x + y)$.
 58. $a^3 - b^3 - a(a^2 - b^2) + b(a - b)^2$.
 59. $9x^2y^3 - 3xy^5 - 6y^4$. 60. $6x^3 + 13xy + 6y^2$.
 61. $6a^3b^3 - ab^3 - 12b^4$.
 62. $a^2 + 2ad + d^2 - 4b^2 + 12bc - 9c^2$.
 63. $x^3 - 2x^2y + 4xy^2 - 8y^3$. 64. $4a^3x^3 - 8abx + 3b^3$.

65. $18x^2 - 24xy + 8y^2 + 9x - 6y$. 74. $16a^2x - 2x^4$.
 66. $2x^2 + 2xy - 12y^2 + 6xz + 18yz$. 75. $32bx^2 - 4by^2$.
 67. $(x+y)^2 - 1 - xy(x+y+1)$. 76. $x - 27x^4$.
 68. $x^3 - y^3 - z^3 + 2yz + x + y - z$. 77. $x^{12} - y^{12}$.
 69. $2x^2 + 4xy + 2y^2 + 2ax + 2ay$. 78. $49m^2 - 121n^2$.
 70. $16a^2b + 32abc + 12bc^2$. 79. $16 - 81y^4$.
 71. $m^2p - m^2q - n^2p + n^2q$. 80. $12z^4 - z^2 - 6$.
 72. $12ax^2 - 14axy - 6ay^2$. 81. $x^3 - x^2 + x - 1$.
 73. $2x^3 + 4x^2 - 70x$. 82. $x^2 + 2x + 1 - y^2$.
 83. $49(a-b)^2 - 64(m-n)^2$.
 84. $1 - \left(\frac{a^2 + b^2 - c^2}{2ab} \right)^2$.
 85. $x^2 - 53x + 360$.
 86. $x^3 - 2x^2y + x^2 - 4x + 8y - 4$.
 87. $2ab - 2bc - ae + ce + 2b^2 - be$.
 88. $125x^5 + 350x^3y^2 + 245xy^4$.
 89. $a^6 + a^5b + a^4b^2 + a^3b^3 + a^2b^4 + ab^5$.
 90. $2a^4x - 2a^3cx + 2ac^3x - 2c^4x$.
 91. $6x^2 - 5xy - 6y^2 + 3xz + 15yz - 9z^2$.
 92. $4x^2 - 9xy + 2y^2 - 3xz - yz - z^2$.
 93. $3a^2 - 7ab + 2b^2 + 5ac - 5bc + 2c^2$.
 94. $x^4 - 2x^3 + x^2 - 8x + 8$.
 95. $5x^2 - 8xy + 3y^2 - 5x + 3y$.
 96. $a^2 - 2ad + d^2 - 4b^2 + 12bc - 9c^2$.
 97. $(x^2 - x - 6)(x^2 - x - 20)$.

CHAPTER VII.

COMMON FACTORS AND MULTIPLES.

142. A **common factor** of two or more expressions is an expression which is contained in each of them without a remainder. Thus,

$5a$ is a common factor of $20a$ and $25a$;

$3x^2y^2$ is a common factor of $12x^2y^2$ and $15x^3y^3$.

143. Two expressions which have no common factor except 1, are said to be *prime* to each other.

144. The **Highest Common Factor** of two or more expressions is the product of all the factors common to the expressions.

Thus, $3a^2$ is the highest common factor of $3a^2$, $6a^3$, and $12a^4$.

$5x^2y^2$ is the highest common factor of $10x^3y^2$ and $15x^2y^3$.

For brevity, H. C. F. will be used for Highest Common Factor.

(1) Find the H. C. F. of $42a^3b^2x$ and $21a^2b^3x^2$.

$$42a^3b^2x = 2 \times 3 \times 7 \times a^3 \times b^2 \times x;$$

$$21a^2b^3x^2 = 3 \times 7 \times a^2 \times b^3 \times x^2.$$

$$\begin{aligned}\therefore \text{the H. C. F.} &= 3 \times 7 \times a^2 \times b^2 \times x \\ &= 21a^2b^2x.\end{aligned}$$

(2) Find the H. C. F. of $2a^2x + 2ax^2$ and $3abxy + 3bx^2y$.

$$2a^2x + 2ax^2 = 2ax(a + x);$$

$$3abxy + 3bx^2y = 3bxy(a + x).$$

$$\therefore \text{the H. C. F.} = x(a + x).$$

(3) Find the H. C. F. of

$$8a^2x^2 - 24a^2x + 16a^2 \text{ and } 12ax^2y - 12axy - 24ay.$$

$$\begin{aligned} 8a^2x^2 - 24a^2x + 16a^2 &= 8a^2(x^2 - 3x + 2), \\ &= 2^3a^2(x-1)(x-2); \end{aligned}$$

$$\begin{aligned} 12ax^2y - 12axy - 24ay &= 12ay(x^2 - x - 2), \\ &= 2^3 \times 3ay(x+1)(x-2). \end{aligned}$$

$$\begin{aligned} \therefore \text{the H. C. F.} &= 2^2a(x-2), \\ &= 4a(x-2). \end{aligned}$$

Hence, to find the H. C. F. of two or more expressions :

Resolve each expression into its lowest factors.

Select from these the lowest power of each common factor, and find the product of these powers.

EXERCISE 47.

Find the H. C. F. of :

1. $18ab^2c^2d$ and $36a^2bcd^2$. 2. $17pq^3$, $34p^2q$, and $51p^3q^3$.

3. $8x^3y^2z^4$, $12x^2y^3z^3$, and $20x^4y^3z^3$.

4. $30x^4y^3$, $90x^2y^3$, and $120x^3y^4$.

5. $a^2 - b^2$ and $a^2 - b^3$. 7. $a^3 + x^3$ and $(a+x)^3$.

6. $a^2 - x^2$ and $(a-x)^2$. 8. $9x^2 - 1$ and $(3x+1)^2$.

9. $7x^2 - 4x$ and $7a^2x - 4a^3$.

10. $12a^3x^2y - 4a^2xy^2$ and $30a^2x^3y^2 - 10a^2x^2y^3$.

11. $8a^3b^2c - 12a^2bc^3$ and $6ab^4c + 4ab^3c^2$.

12. $x^2 - 2x - 3$ and $x^2 + x - 12$.

13. $2a^3 - 2ab^2$ and $4b(a+b)^2$.

14. $12x^2y(x-y)(x-3y)$ and $18x^2(x-y)(3x-y)$.

15. $3x^3 + 6x^2 - 24x$ and $6x^3 - 96x$.

16. $ac(a-b)(a-c)$ and $bc(b-a)(b-c)$.
17. $10x^3y - 60x^2y^2 + 5xy^3$ and $5x^2y^2 - 5xy^3 - 100y^4$.
18. $x(x+1)^2$, $x^2(x^2-1)$, and $2x(x^2-x-2)$.
19. $3x^2-6x+3$, $6x^2+6x-12$, and $12x^2-12$.
20. $6(a-b)^4$, $8(a^2-b^2)^2$, and $10(a^4-b^4)$.
21. x^2-y^2 , $(x+y)^2$, and $x^2+3xy+2y^2$.
22. x^2-y^2 , x^3-y^3 , and $x^2-7xy+6y^2$.
23. x^2-1 , x^3-1 , and x^2+x-2 .

145. When it is required to find the H. C. F. of two or more expressions which cannot readily be resolved into their factors, the method to be employed is similar to that of the corresponding case in arithmetic. And as that method consists in obtaining pairs of continually decreasing numbers which contain as a factor the H. C. F. required; so in algebra, pairs of expressions of continually decreasing degrees are obtained, which contain as a factor the H. C. F. required.

The method depends upon two principles:

1. *Any factor of an expression is a factor also of any multiple of that expression.*

Thus, if F represent a factor of an expression A , so that $A = nF$, then $mA = mnF$. That is, mA contains the factor F .

2. *Any common factor of two expressions is a factor of the sum or difference of any multiples of the expressions.*

Thus, if F represent a common factor of the expressions A and B so that

$$\begin{aligned}
 & A = mF, \text{ and } B = nF; \\
 \text{then} \quad & pA = pmF, \text{ and } qB = qnF. \\
 \text{Hence,} \quad & pA \pm qB = pmF \pm qnF, \\
 & = (pm \pm qn)F. \\
 \text{That is,} \quad & pA \pm qB \text{ contains the factor } F.
 \end{aligned}$$

146. The general proof of this method as applied to *numbers* is as follows:

Let a and b be two numbers, of which a is the greater. The operation may be represented by:

$$\begin{array}{rcl}
 b) \ a \ (p & 42) \ 154 \ (3 & nF) \ mF \ (p \\
 \underline{pb} & \underline{126} & \underline{pnF} \\
 c) \ b \ (q & 28) \ 42 \ (1 & \underline{cF}) \ nF \ (q \\
 \underline{qc} & \underline{28} & \underline{qcF} \\
 d) \ c \ (r & 14) \ 28 \ (2 & \underline{F}) \ cF \ (c \\
 \underline{rd} & \underline{28} & \underline{cF}
 \end{array}$$

p , q , and r represent the several quotients,

c and d represent the remainders,

and d is supposed to be contained exactly in c .

The numbers represented are all integral.

Then

$$c = rd,$$

$$b = qc + d = qrd + d = (qr + 1)d,$$

$$\begin{aligned}
 a &= pb + c = pqr d + pd + rd, \\
 &= (pqr + p + r)d.
 \end{aligned}$$

$\therefore d$ is a common factor of a and b .

It remains to show that d is the *highest* common factor of a and b .

Let f represent the highest common factor of a and b .

Now $c = a - pb$, and f is a common factor of a and b .

\therefore by (2) f is a factor of c .

Also, $d = b - qc$, and f is a common factor of b and c .

\therefore by (2) f is a factor of d .

That is, d contains the *highest common factor* of a and b .

But it has been shown that d is a common factor of a and b .

$\therefore d$ is the *highest common factor* of a and b .

NOTE. The second operation represents the application of the method to a particular case. The third operation is intended to represent clearly that every remainder in the course of the operation contains as a factor the H. C. F. sought, and that this is the *highest factor common* to that remainder and the preceding divisor.

147. By the same method, find the H. C. F. of

$$2x^3 + x - 3 \text{ and } 4x^3 + 8x^2 - x - 6.$$

$$\begin{array}{r}
 2x^3 + x - 3 \quad 4x^3 + 8x^2 - x - 6 \quad (2x + 3) \\
 \underline{4x^3 + 2x^2 - 6x} \\
 6x^2 + 5x - 6 \\
 \underline{6x^2 + 3x - 9} \\
 2x + 3 \quad 2x^3 + x - 3 \quad (x - 1) \\
 \underline{2x^3 + 3x} \\
 -2x - 3 \\
 \underline{-2x - 3}
 \end{array}$$

\therefore the H. F. C. = $2x + 3$.

The given expressions are arranged according to the descending powers of x .

The expression whose first term is of the lower degree is taken for the divisor; and each division is continued until the first term of the remainder is of lower degree than that of the divisor.

148. This method is of use only to determine the **compound** factor of the H. C. F. **Simple** factors of the given expressions must first be separated from them, and the highest common factor of these must be reserved to be multiplied into the compound factor obtained.

Find the H. C. F. of

$$12x^4 + 30x^3 - 72x^2 \text{ and } 32x^3 + 84x^2 - 176x.$$

$$12x^4 + 30x^3 - 72x^2 = 6x^2(2x^2 + 5x - 12).$$

$$32x^3 + 84x^2 - 176x = 4x(8x^2 + 21x - 44).$$

$$6x^2 \text{ and } 4x \text{ have } 2x \text{ common.}$$

$$\begin{array}{r}
 2x^3 + 5x - 12 \quad 8x^3 + 21x - 44 \quad (4) \\
 \underline{8x^3 + 20x - 48} \\
 x + 4 \quad 2x^3 + 5x - 12 \quad (2x - 3) \\
 \underline{2x^3 + 8x} \\
 -3x - 12 \\
 \underline{-3x - 12}
 \end{array}$$

\therefore the H. F. C. = $2x(x + 4)$.

149. Modifications of this method are sometimes needed.

- (1) Find the H. C. F. of $4x^2 - 8x - 5$ and $12x^2 - 4x - 65$.

$$\begin{array}{r} 4x^2 - 8x - 5 \quad 12x^2 - 4x - 65 \quad (3) \\ \underline{12x^2 - 24x - 15} \\ 20x - 50 \end{array}$$

The first division ends here, for $20x$ is of lower degree than $4x^2$. But if $20x - 50$ be made the divisor, $4x^2$ will not contain $20x$ an *integral* number of times.

Now, it is to be remembered that the H. C. F. sought is *contained in the remainder* $20x - 50$, and that it is a *compound factor*. Hence if the *simple factor* 10 be removed, the H. C. F. must still be contained in $2x - 5$, and therefore the process may be continued with $2x - 5$ for a divisor.

$$\begin{array}{r} 2x - 5 \quad 4x^2 - 8x - 5 \quad (2x + 1) \\ \underline{4x^2 - 10x} \\ 2x - 5 \\ \underline{2x - 5} \end{array}$$

\therefore the H. C. F. = $2x - 5$.

- (2) Find the H. C. F. of

$$21x^3 - 4x^2 - 15x - 2 \text{ and } 21x^3 - 32x^2 - 54x - 7.$$

$$\begin{array}{r} 21x^3 - 4x^2 - 15x - 2 \quad 21x^3 - 32x^2 - 54x - 7 \quad (1) \\ \underline{21x^3 - 4x^2 - 15x - 2} \\ -28x^2 - 39x - 5 \end{array}$$

The difficulty here cannot be obviated by *removing* a simple factor from the remainder, for $-28x^2 - 39x - 5$ has no simple factor. In this case, the expression $21x^3 - 4x^2 - 15x - 2$ must be *multiplied* by the simple factor 4 to make its first term divisible by $-28x^2$.

The *introduction* of such a factor can in no way affect the H. C. F. sought; for the H. C. F. contains only factors *common to the remainder and the last divisor*, and 4 is not a factor of the remainder.

The *signs* of all the terms of the remainder may be changed; for if an expression A is divisible by $-F$, it is divisible by $+F$.

The process then is continued by changing the signs of the remainder and multiplying the divisor by 4.

$$\begin{array}{r}
 28x^2 + 39x + 5 \quad 84x^2 - 16x^2 - 60x - 8 \quad (3x. \\
 \quad \quad \quad 84x^2 + 117x^2 + 15x \\
 \hline
 \quad \quad \quad -133x^2 - 75x - 8 \\
 \text{Multiply by } -4, \quad \quad \quad -4 \\
 \quad \quad \quad \hline
 \quad \quad \quad 532x^2 + 300x + 32 \quad (19 \\
 \quad \quad \quad 532x^2 + 741x + 95 \\
 \hline
 \text{Divide by } -63, \quad \quad \quad -63 \quad \hline
 \quad \quad \quad 7x + 1
 \end{array}$$

$$\begin{array}{r}
 7x + 1 \quad 28x^2 + 39x + 5 \quad (4x + 5 \\
 \quad \quad \quad 28x^2 + 4x \\
 \hline
 \quad \quad \quad 35x + 5
 \end{array}$$

$$\therefore \text{ the H. C. F. } = 7x + 1. \quad \underline{35x + 5}$$

(3) Find the H. C. F. of

$$8x^3 + 2x - 3 \text{ and } 6x^3 + 5x^2 - 2.$$

$$\begin{array}{r}
 \quad \quad \quad 6x^3 + 5x^2 - 2 \\
 \text{Multiply by 4,} \quad \quad \quad 4 \\
 \hline
 8x^3 + 2x - 3 \quad 24x^3 + 20x^2 - 8 \quad (3x + 7 \\
 \quad \quad \quad 24x^3 + 6x^2 - 9x \\
 \hline
 \quad \quad \quad 14x^2 + 9x - 8 \\
 \text{Multiply by 4,} \quad \quad \quad 4 \\
 \quad \quad \quad \hline
 \quad \quad \quad 56x^2 + 36x - 32 \\
 \quad \quad \quad 56x^2 + 14x - 21 \\
 \hline
 \text{Divide by 11,} \quad \quad \quad 11 \quad \hline
 \quad \quad \quad 2x - 1 \quad 8x^3 + 2x - 3 \quad (4x + 3 \\
 \quad \quad \quad \quad \quad \quad 8x^2 - 4x \\
 \quad \quad \quad \quad \quad \quad \hline
 \quad \quad \quad \quad \quad \quad 6x - 3 \\
 \therefore \text{ the H. C. F. } = 2x - 1. \quad \quad \quad \underline{6x - 3}
 \end{array}$$

In this case it is necessary to multiply by 4 the *given expression* $6x^3 + 5x^2 - 2$ to make its first term divisible by $8x^3$, 4 being obviously not a *common factor*.

The following arrangement of the work will be found most convenient :

$8x^2 + 2x - 3$	$6x^3 + 5x^2 - 2$	
$8x^2 - 4x$	4	
$\hline 6x - 3$	$24x^3 + 20x^2 - 8$	$3x$
$\hline 6x - 3$	$24x^3 + 6x^2 - 9x$	
	$\hline 14x^2 + 9x - 8$	
	4	
	$\hline 56x^2 + 36x - 32$	$+ 7$
	$\hline 56x^2 + 14x - 21$	
	$\hline 11 \overline{) 22x - 11}$	
	$\hline 2x - 1$	$4x + 3$

150. From the foregoing examples it will be seen that, in the algebraic process of finding the highest common factor, the following steps, in the order here given, must be carefully observed :

I. Simple factors of the given expressions are to be removed from them, and the highest common factor of these is to be reserved as a factor of the H. C. F. sought.

II. The resulting compound expressions are to be arranged according to the *descending* powers of a common letter ; and that expression which is of the lower degree is to be taken for the divisor ; or, if both are of the same degree, that whose first term has the smaller coefficient.

III. Each division is to be continued until the remainder is of lower degree than the divisor.

IV. If the final remainder of any division is found to contain a factor that is not a *common* factor of the given expressions, *this factor is to be removed* ; and the resulting expression is to be used as the next divisor.

V. A dividend whose first term is not exactly divisible by the first term of the divisor, is to be *multiplied* by such an expression as will make it thus divisible.

EXERCISE 48.

Find the H. C. F. of :

1. $5x^2 + 4x - 1$, $20x^2 + 21x - 5$.
2. $2x^3 - 4x^2 - 13x - 7$, $6x^3 - 11x^2 - 37x - 20$.
3. $6a^4 + 25a^3 - 21a^2 + 4a$, $24a^4 + 112a^3 - 94a^2 + 18a$.
4. $9x^3 + 9x^2 - 4x - 4$, $45x^3 + 54x^2 - 20x - 24$.
5. $27x^6 - 3x^4 + 6x^3 - 3x^2$, $162x^6 + 48x^3 - 18x^2 + 6x$.
6. $20x^3 - 60x^2 + 50x - 20$, $32x^4 - 92x^3 + 68x^2 - 24x$.
7. $4x^2 - 8x - 5$, $12x^2 - 4x - 65$.
8. $3a^3 - 5a^2x - 2ax^2$, $9a^3 - 8a^2x - 20ax^2$.
9. $10x^3 + x^2 - 9x + 24$, $20x^4 - 17x^2 + 48x - 3$.
10. $8x^3 - 4x^2 - 32x - 182$, $36x^3 - 84x^2 - 111x - 126$.
11. $5x^2(12x^3 + 4x^2 + 17x - 3)$, $10x(24x^3 - 52x^2 + 14x - 1)$.
12. $9x^4y - x^2y^3 - 20xy^4$, $18x^3y - 18x^2y^2 - 2xy^3 - 8y^4$.
13. $6x^2 - x - 15$, $9x^2 - 3x - 20$.
14. $12x^3 - 9x^2 + 5x + 2$, $24x^2 + 10x + 1$.
15. $6x^3 + 15x^2 - 6x + 9$, $9x^3 + 6x^2 - 51x + 36$.
16. $4x^3 - x^2y - xy^2 - 5y^3$, $7x^3 + 4x^2y + 4xy^2 - 3y^3$.
17. $2a^3 - 2a^2 - 3a - 2$, $3a^3 - a^2 - 2a - 16$.
18. $12y^3 + 2y^2 - 94y - 60$, $48y^3 - 24y^2 - 348y + 30$.
19. $9x(2x^4 - 6x^3 - x^2 + 15x - 10)$,
 $6x^2(4x^4 + 6x^3 - 4x^2 - 15x - 15)$.
20. $15x^4 + 2x^3 - 75x^2 + 5x + 2$, $35x^4 + x^3 - 175x^2 + 30x + 1$.
21. $21x^4 - 4x^3 - 15x^2 - 2x$, $21x^3 - 32x^2 - 54x - 7$.
22. $9x^4y - 22x^2y^3 - 3xy^4 + 10y^5$, $9x^5y - 6x^4y^2 + x^3y^3 - 25xy^4$.

23. $6x^5 - 4x^4 - 11x^3 - 3x^2 - 3x - 1$,
 $4x^4 + 2x^3 - 18x^2 + 3x - 5$.
24. $x^4 - ax^3 - a^2x^2 - a^3x - 2a^4$, $3x^3 - 7ax^2 + 3a^2x - 2a^3$.

151. The H. C. F. of three expressions will be obtained by finding the H. C. F. of two of them, and then of that and the third expression.

For, if A , B , and C are three expressions,
 and D the highest common factor of A and B ,
 and E the highest common factor of D and C ,
 then D contains every factor common to A and B ,
 and E contains every factor common to D and C .
 $\therefore E$ contains every factor common to A , B , and C .

EXERCISE 49.

Find the H. C. F. of :

- $2x^2 + x - 1$, $x^2 + 5x + 4$, $x^3 + 1$.
- $y^3 - y^2 - y + 1$, $3y^2 - 2y - 1$, $y^3 - y^2 + y - 1$.
- $x^3 - 4x^2 + 9x - 10$, $x^3 + 2x^2 - 3x + 20$, $x^3 + 5x^2 - 9x + 35$.
- $x^3 - 7x^2 + 16x - 12$, $3x^3 - 14x^2 + 16x$,
 $5x^3 - 10x^2 + 7x - 14$.
- $y^3 - 5y^2 + 11y - 15$, $y^3 - y^2 + 3y + 5$,
 $2y^3 - 7y^2 + 16y - 15$.
- $2x^3 + 3x - 5$, $3x^3 - x - 2$, $2x^3 + x - 3$.
- $x^3 - 1$, $x^3 - x^2 - x - 2$, $2x^3 - x^2 - x - 3$.
- $x^3 - 3x - 2$, $2x^3 + 3x^2 - 1$, $x^3 + 1$.
- $12(x^4 - y^4)$, $10(x^6 - y^6)$, $8(x^4y + xy^4)$.
- $x^4 + xy^3$, $x^3y + y^4$, $x^4 + x^2y^2 + y^4$.
- $2(x^3y - xy^3)$, $3(x^3y - xy^3)$, $4(x^4y - xy^4)$, $5(x^5y - xy^5)$.

LOWEST COMMON MULTIPLE.

152. A common multiple of two or more expressions is an expression which is exactly divisible by each of them.

153. The **Lowest Common Multiple** of two or more expressions is the product of all the factors of the expressions, each factor being written with its highest exponent.

154. The lowest common multiple of two expressions which have no common factor will be their product.

For brevity L. C. M. will be used for Lowest Common Multiple.

(1) Find the L. C. M. of $12a^2c$, $14bc^2$, $36ab^2$.

$$12a^2c = 2^2 \times 3a^2c,$$

$$14bc^2 = 2 \times 7bc^2,$$

$$36ab^2 = 2^2 \times 3^2ab^2.$$

$$\therefore \text{The L. C. M.} = 2^2 \times 3^2 \times 7a^2b^2c^2 = 252a^2b^2c^2.$$

(2) Find the L. C. M. of

$$2a^2 + 2ax, \quad 6a^2 - 6x^2, \quad 3a^2 - 6ax + 3x^2.$$

$$2a^2 + 2ax = 2a(a+x),$$

$$6a^2 - 6x^2 = 2 \times 3(a+x)(a-x),$$

$$3a^2 - 6ax + 3x^2 = 3(a-x)^2.$$

$$\therefore \text{the L. C. M.} = 6a(a+x)(a-x)^2.$$

EXERCISE 50.

Find the L. C. M. of

1. $4a^3x$, $6a^2x^2$, $2ax^2$.

4. $x^2 - 1$, $x^2 - x$.

2. $18ax^2$, $72ay^2$, $12xy$.

5. $a^2 - b^2$, $a^2 + ab$.

3. x^2 , $ax + x^2$.

6. $2x - 1$, $4x^2 - 1$.

7. $a+b$, a^3+b^3 .
8. x^3-1 , x^2+1 , x^4-1 .
9. x^3-x , x^3-1 , x^3+1 .
10. x^3-1 , x^3-x , x^3-1 .
11. $2a+1$, $4a^2-1$, $8a^3+1$.
12. $(a+b)^2$, a^2-b^2 .
13. $4(1+x)$, $4(1-x)$, $2(1-x^2)$.
14. $x-1$, x^2+x+1 , x^3-1 .
15. x^3-y^3 , $(x+y)^3$, $(x-y)^3$.
16. x^3-y^3 , $3(x-y)^3$, $12(x^3+y^3)$.
17. $6(x^2+xy)$, $8(xy-y^2)$, $10(x^2-y^2)$.
18. x^2+5x+6 , x^2+6x+8 .
19. a^2-a-20 , a^2+a-12 .
20. $x^2+11x+30$, $x^2+12x+35$.
21. $x^2-9x-22$, $x^2-13x+22$.
22. $4ab(a^2-3ab+2b^2)$, $5a^2(a^2+ab-6b^2)$.
23. $20(x^2-1)$, $24(x^2-x-2)$, $16(x^2+x-2)$.
24. $12xy(x^2-y^2)$, $2x^2(x+y)^2$, $3y^2(x-y)^2$.
25. $(a-b)(b-c)$, $(b-c)(c-a)$, $(c-a)(a-b)$.
26. $(a-b)(a-c)$, $(b-a)(b-c)$, $(c-a)(c-b)$.
27. x^5-4x^3+3x , $x^4+x^3-12x^2$, $x^5+3x^4-4x^3$.
28. x^2y-xy^2 , $3x(x-y)^2$, $4y(x-y)^3$.
29. $(a+b)^2-(c+d)^2$, $(a+c)^2-(b+d)^2$, $(a+d)^2-(b+c)^2$.
30. $(2x-4)(3x-6)$, $(x-3)(4x-8)$, $(2x-6)(5x-10)$.

155. When the expressions cannot be readily resolved into their factors, the expressions may be resolved by finding their H. C. F.

Find the L. C. M. of

$$6x^3 - 11x^2y + 2y^3 \text{ and } 9x^3 - 22xy^2 - 8y^3.$$

$6x^3 - 11x^2y + 2y^3$	$+ 2y^3$	$9x^3 - 22xy^2 - 8y^3$	3
$6x^3 - 8x^2y - 4xy^2$		2	
$- 3x^2y + 4xy^2 + 2y^3$		$18x^3 - 44xy^2 - 16y^3$	
$- 3x^2y + 4xy^2 + 2y^3$		$18x^3 - 33x^2y + 6y^3$	
		$11y) 33x^2y - 44xy^2 - 22y^3$	
		$3x^2 - 4xy - 2y^2$	$2x - y$

Hence, $6x^3 - 11x^2y + 2y^3 = (2x - y)(3x^2 - 4xy - 2y^2)$,
and $9x^3 - 22xy^2 - 8y^3 = (3x + 4y)(3x^2 - 4xy - 2y^2)$.

\therefore the L. C. M. $= (2x - y)(3x + 4y)(3x^2 - 4xy - 2y^2)$.

In this example we find the H. C. F. of the given expressions, and divide each of them by the H. C. F.

156. It will be observed that the product of the H. C. F. and the L. C. M. of two expressions is equal to the product of the given expressions. For,

Let A and B denote the two expressions, and D their H. C. F.

Suppose $A = aD$, and $B = bD$;

Since D consists of all the factors common to A and B , a and b have no common factor.

\therefore L. C. M. of a and b is ab .

Hence, the L. C. M. of aD and bD is abD .

Now, $A = aD$, and $B = bD$;

$\therefore AB = abD \times D$.

$\therefore \frac{AB}{D} = abD =$ the lowest common multiple. That is,

The L. C. M. of two expressions can be found by dividing their product by their H. C. F.

Or, by dividing one of the expressions by the H. C. F., and multiplying the result by the other expression.

157. To find the L. C. M. of *three* expressions A , B , C . Find M , the L. C. M. of A and B ; then the L. C. M. of M and C is the L. C. M. required.

EXERCISE 51.

Find the L. C. M. of :

1. $6x^2 - x - 2$, $21x^2 - 17x + 2$, $14x^2 + 5x - 1$.
2. $x^2 - 1$, $x^2 + 2x - 3$, $6x^2 - x - 2$.
3. $x^3 - 27$, $x^2 - 15x + 36$, $x^3 - 3x^2 - 2x + 6$.
4. $5x^2 + 19x - 4$, $10x^2 + 13x - 3$.
5. $12x^2 + xy - 6y^2$, $18x^2 + 18xy - 20y^2$.
6. $x^4 - 2x^3 + x$, $2x^4 - 2x^3 - 2x - 2$.
7. $12x^2 + 2x - 4$, $12x^2 - 42x - 24$, $12x^2 - 28x - 24$.
8. $x^3 - 6x^2 + 11x - 6$, $x^3 - 9x^2 + 26x - 24$,
 $x^3 - 8x^2 + 19x - 12$.
9. $x^3 - 4a^2$, $x^3 + 2ax^2 + 4a^2x + 8a^3$,
 $x^3 - 2ax^2 + 4a^2x - 8a^3$.
10. $x^3 + 2x^2y - xy^2 - 2y^3$, $x^3 - 2x^2y - xy^2 + 2y^3$.
11. $1 + p + p^2$, $1 - p + p^2$, $1 + p^2 + p^4$.
12. $(1 - a)$, $(1 - a)^2$, $(1 - a)^3$.
13. $(a + c)^2 - b^2$, $(a + b)^2 - c^2$, $(b + c)^2 - a^2$.
14. $3c^3 - 3c^2y + cy^2 - y^3$, $4c^3 - c^2y - 3cy^2$.
15. $m^3 - 8m + 3$, $m^6 + 3m^5 + m + 3$.
16. $20n^4 + n^2 - 1$, $25n^4 + 5n^3 - n - 1$.
17. $b^4 - 2b^3 + b^2 - 8b + 8$, $4b^3 - 12b^2 + 9b - 1$.
18. $2r^5 - 8r^4 + 12r^3 - 8r^2 + 2r$, $3r^5 - 6r^3 + 3r$.

CHAPTER VIII.

FRACTIONS.

158. IN Arithmetic, an expression of the form of $\frac{4}{5}$ is employed to indicate that 4 units are divided into 5 equal parts, and that *one* of these parts is taken ; or, that *one* unit is divided into 5 equal parts, and that 4 of these parts are taken.

The expression is also used to indicate *the quotient of 4 divided by 5*.

159. The expression $\frac{4}{5}$ is called a **fraction**. 4 is the **numerator**, and 5 the **denominator**.

160. The numerator and denominator together are called the **terms** of the fraction.

161. The denominator shows into how many equal parts the unit is divided and therefore *names* the part ; and the numerator shows the number of the parts taken.

It will be observed that a figure written *above* the line in a fraction serves a very different purpose from that of a figure written *below* the line.

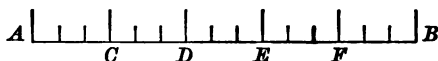
A figure written above the line denotes **number** ;

A figure written below the line denotes **name**.

162. Every whole number may be written in the form of a fraction with unity for its denominator ; thus, $4 = \frac{4}{1}$.

TO REDUCE A FRACTION TO ITS LOWEST TERMS.

163. Let the line AB be divided into 5 equal parts, at the points C, D, E, F .



Then AF is $\frac{4}{5}$ of AB . (1)

Now let each of the parts be subdivided into 3 equal parts.

Then AB contains 15 of these subdivisions, and AF contains 12 of these subdivisions.

$\therefore AF$ is $\frac{4}{5}$ of AB . (2)

Comparing (1) and (2), it is evident that $\frac{4}{5} = \frac{4}{5}$.

In general :

If we suppose AB to be divided into b equal parts, and that AF contains a of these parts, where a and b signify *positive whole numbers*,

Then AF is $\frac{a}{b}$ of AB . (3)

Now, if we suppose each of the parts to be subdivided into c equal parts, c being a positive whole number,

Then AB contains bc of these subdivisions, and AF contains ac of these subdivisions.

$\therefore AF$ is $\frac{ac}{bc}$ of AB . (4)

Comparing (3) and (4), it is evident that

$$\frac{a}{b} = \frac{ac}{bc}.$$

Since $\frac{ac}{bc}$ is obtained by multiplying by c both terms of the fraction $\frac{a}{b}$,

and, conversely, $\frac{a}{b}$ is obtained by dividing by c both terms of the fraction $\frac{ac}{bc}$, it follows that

If the numerator and denominator of a fraction be multiplied by the same number, or divided by the same number, the value of the fraction is not altered.

In Algebra, a fraction simply indicates *the quotient of the numerator divided by the denominator*, but the laws that apply to fractions in Arithmetic apply also to fractions in Algebra.

164. To reduce a fraction to lower terms,

Divide the numerator and denominator by any common factor.

A fraction is expressed in its lowest terms when both numerator and denominator are divided by their H. C. F.

Reduce the following fractions to their lowest terms :

$$(1) \frac{a^3 - x^3}{a^2 - x^2} = \frac{(a-x)(a^2 + ax + x^2)}{(a-x)(a+x)} = \frac{a^2 + ax + x^2}{a+x}.$$

$$(2) \frac{a^2 + 7a + 10}{a^2 + 5a + 6} = \frac{(a+5)(a+2)}{(a+3)(a+2)} = \frac{a+5}{a+3}.$$

$$(3) \frac{6x^2 - 5x - 6}{8x^2 - 2x - 15} = \frac{(2x-3)(3x+2)}{(2x-3)(4x+5)} = \frac{3x+2}{4x+5}.$$

$$(4) \frac{a^3 - 7a^2 + 16a - 12}{3a^3 - 14a^2 + 16a}.$$

Since in Ex. (4) no common factor can be determined by inspection, it is necessary to find the H. C. F. of the numerator and denominator by the method of division.

Suppress the factor a of the denominator and proceed to divide :

$$\begin{array}{r|l|l}
 a^3 - 7a^2 + 16a - 12 & 3a^2 - 14a + 16 & a - 7 \\
 3 & 3a^2 - 6a & \\
 \hline
 3a^3 - 21a^2 + 48a - 36 & - 8a + 16 & 3a - 8 \\
 3a^3 - 14a^2 + 16a & \hline
 - 7a^2 + 32a - 36 & - 8a + 16 & \\
 3 & & \\
 \hline
 - 21a^2 + 96a - 108 & & \\
 - 21a^2 + 98a - 112 & & \\
 \hline
 - 2 \overline{) - 2a + 4} & & \\
 a - 2 & &
 \end{array}$$

\therefore the H. C. F. = $a - 2$.

Now, if $a^3 - 7a^2 + 16a - 12$ be divided by $a - 2$, the result is $a^2 - 5a + 6$; and if $3a^3 - 14a^2 + 16a$ be divided by $a - 2$, the result is $3a^2 - 8a$.

$$\therefore \frac{a^3 - 7a^2 + 16a - 12}{3a^3 - 14a^2 + 16a} = \frac{a^2 - 5a + 6}{3a^2 - 8a}.$$

165. If the common factors cannot be determined by inspection, the H. C. F. must be found by division.

EXERCISE 52.

Reduce to lowest terms :

1. $\frac{x^2 - 1}{4x(x + 1)}$

6. $\frac{a^3 + 1}{a^3 + 2a^2 + 2a + 1}$

2. $\frac{x^2 - 9x + 20}{x^2 - 7x + 12}$

7. $\frac{a^2 - a - 20}{a^2 + a - 12}$

3. $\frac{x^2 - 2x - 3}{x^2 - 10x + 21}$

8. $\frac{x^3 - 4x^2 + 9x - 10}{x^3 + 2x^2 - 3x + 20}$

4. $\frac{x^4 + x^2 + 1}{x^2 + x + 1}$

9. $\frac{x^3 - 5x^2 + 11x - 15}{x^3 - x^2 + 3x + 5}$

5. $\frac{x^6 + 2x^3y^3 + y^6}{x^6 - y^6}$

10. $\frac{x^4 + x^3y + xy^3 - y^4}{x^4 - x^3y - xy^3 - y^4}$

- | | |
|---|---|
| 11. $\frac{a^3+4a^2-5}{a^3-3a+2}$ | 21. $\frac{a^5-a^4b-ab^4+b^5}{a^4-a^3b-a^2b^2+ab^3}$ |
| 12. $\frac{3x^3+2x-1}{x^3+x^2-x-1}$ | 22. $\frac{(a+b)^3}{a^3-ab-2b^2}$ |
| 13. $\frac{x^3-3x^2+4x-2}{x^3-x^2-2x+2}$ | 23. $\frac{3ab(a^2-b^2)}{4(a^3b-ab^3)^2}$ |
| 14. $\frac{4x^3-12ax+9a^2}{8x^3-27a^3}$ | 24. $\frac{a^3+2ab+b^2-c^2}{a^2+ab-ac}$ |
| 15. $\frac{15a^3+ab-2b^3}{9a^3+3ab-2b^3}$ | 25. $\frac{6x^3-11x^2y+3xy^3}{6x^2y-5xy^2-6y^3}$ |
| 16. $\frac{a^3-b^3-2bc-c^2}{a^3+2ab+b^2-c^2}$ | 26. $\frac{a^2-(b+c+d)^2}{(a-b)^2-(c+d)^2}$ |
| 17. $\frac{x^4-x^2-2x+2}{2x^3-x-1}$ | 27. $\frac{6x^2-5x-6}{8x^2-2x-15}$ |
| 18. $\frac{x^3-6x^2+11x-6}{x^3-2x^2-x+2}$ | 28. $\frac{x^4+x^2y^2+y^4}{(x-y)(x^3-y^3)}$ |
| 19. $\frac{6x^3-23x^2+16x-3}{6x^3-17x^2+11x-2}$ | 29. $\frac{x^5+y^5}{x^4-x^2y^2+y^4}$ |
| 20. $\frac{x^4-x^3-x+1}{x^4-2x^3-x^2-2x+1}$ | 30. $\frac{(a^3+b^3)(a^2+ab+b^2)}{(a^3-b^3)(a^2-ab+b^2)}$ |

TO REDUCE A FRACTION TO AN INTEGRAL OR MIXED
EXPRESSION.

Change $\frac{x^3+1}{x-1}$ to a mixed expression.

$$(x^3+1) \div (x-1) = x^2+x+1 + \frac{2}{x-1}. \quad \text{Hence,}$$

166. *If the degree of the numerator of a fraction equals or exceeds that of the denominator, the fraction may be changed to the form of a mixed or integral expression by dividing the numerator by the denominator.*

If there is a remainder, this remainder must be written as the numerator of a fraction of which the divisor is the denominator, and this fraction with its proper sign must be annexed to the integral part of the quotient.

EXERCISE 53.

Change to integral or mixed expressions :

$$1. \frac{x^3 - 2x + 1}{x - 1}.$$

$$6. \frac{10a^2 - 17ax + 10x^2}{5a - x}.$$

$$2. \frac{3x^2 + 2x + 1}{x + 4}.$$

$$7. \frac{16(3x^2 + 1)}{4x - 1}.$$

$$3. \frac{3x^2 + 6x + 5}{x + 4}.$$

$$8. \frac{2x^2 - 5x - 2}{x - 4}.$$

$$4. \frac{a^3 - ax + x^3}{a + x}.$$

$$9. \frac{a^3 + b^3}{a - b}.$$

$$5. \frac{2x^2 + 5}{x - 3}.$$

$$10. \frac{5x^3 - x^2 + 5}{5x^2 + 4x - 1}.$$

TO REDUCE A MIXED EXPRESSION TO THE FORM OF A FRACTION.

167. In arithmetic $5\frac{3}{4}$ means $5 + \frac{3}{4}$.

But in algebra the fraction connected with the integral expression, as well as the integral expression, may be positive or negative; so that a mixed expression may occur in any one of the following forms :

$$n + \frac{a}{b}; \quad n - \frac{a}{b}; \quad -n + \frac{a}{b}; \quad -n - \frac{a}{b}.$$

Change $n + \frac{a}{b}$ to a fractional form.

Since when b is a positive integral number there are b *bths* in 1, in n there will be n times b *bths*, that is, nb *bths*, which, with the additional a *bths*, make $(nb + a)$ *bths*.

$$\therefore n + \frac{a}{b} = \frac{nb + a}{b}.$$

In like manner :

$$n - \frac{a}{b} = \frac{nb - a}{b};$$

$$-n + \frac{a}{b} = \frac{-nb + a}{b};$$

and

$$-n - \frac{a}{b} = \frac{-nb - a}{b}.$$

The same law holds whatever be the value of b .

168. Hence, to reduce a mixed expression to a fraction,
Multiply the integral expression by the denominator, to the product annex the numerator, and under the result write the denominator.

169. It will be seen that the *sign* before the fraction is transferred to the *numerator* when the mixed expression is reduced to the fractional form, for the denominator shows only what *part* of the numerator is to be added or subtracted.

The dividing line has the force of a vinculum or parenthesis affecting the numerator; therefore if a *minus sign* precede the dividing line, and this line be removed, the *sign of every term of the numerator must be changed*. Thus,

$$n - \frac{a - b}{c} = \frac{cn - (a - b)}{c} = \frac{cn - a + b}{c}.$$

(1) Change to fractional form $x-1+\frac{x-1}{x}$.

$$\begin{aligned} & x-1+\frac{x-1}{x}, \\ &= \frac{x^2-x+(x-1)}{x}, \\ &= \frac{x^2-x+x-1}{x}, \\ &= \frac{x^2-1}{x}. \end{aligned}$$

(2) Change to fractional form $x-1-\frac{x-1}{x}$.

$$\begin{aligned} & x-1-\frac{x-1}{x}, \\ &= \frac{x^2-x-(x-1)}{x}, \\ &= \frac{x^2-x-x+1}{x}, \\ &= \frac{x^2-2x+1}{x}. \end{aligned}$$

EXERCISE 54.

Change to fractional form :

1. $1-\frac{x-y}{x+y}$

5. $5a-2b-\frac{3a^2-4b^2}{5a-6b}$

2. $1+\frac{x-y}{x+y}$

6. $a+b-\frac{a^2+b^2}{a+b}$

3. $3x-\frac{1+2x^2}{x}$

7. $7a-\frac{2-3a+4a^2}{5-6a}$

4. $a-x+\frac{a^2+x^2}{a-x}$

8. $3x-\frac{5ax-3}{2a}$

9. $\frac{a+b}{a-b} + 1.$ 15. $2a - b - \frac{2ab}{a+b}.$
10. $\frac{a-b}{a+b} - 1.$ 16. $3x - 10 + \frac{41}{x+4}.$
11. $\frac{2x^2}{x+y} - (x+y).$ 17. $x^2 + x + 1 + \frac{2}{x-1}.$
12. $\frac{5a-12x}{4} + 6a + 3x.$ 18. $x^3 - 3x - \frac{3x(3-x)}{x-2}.$
13. $a - 1 + \frac{1}{a+1}.$ 19. $a^2 - 2ax + 4x^2 - \frac{6x^3}{a+2x}.$
14. $x + 5 - \frac{2x-15}{x-3}.$ 20. $x - a + y + \frac{a^2 - ay + y^2}{x+a}.$

LOWEST COMMON DENOMINATOR.

170. To reduce fractions to equivalent fractions having the lowest common denominator :

Reduce $\frac{3x}{4a^2}$, $\frac{2y}{3a}$, and $\frac{5}{6a^3}$ to equivalent fractions having the lowest common denominator.

The L. C. M. of $4a^2$, $3a$, and $6a^3 = 12a^3$.

If both terms of $\frac{3x}{4a^2}$ be multiplied by $3a$, the value of the fraction will not be altered, but the form will be changed to $\frac{9ax}{12a^3}$; if both terms of $\frac{2y}{3a}$ be multiplied by $4a^2$, the equivalent fraction $\frac{8a^2y}{12a^3}$ is obtained; and, if both terms of $\frac{5}{6a^3}$ be multiplied by 2, the equivalent fraction $\frac{10}{12a^3}$ is obtained.

Hence, $\frac{3x}{4a^2}, \frac{2y}{3a}, \frac{5}{6a^3}$

are equal to $\frac{9ax}{12a^3}, \frac{8a^2y}{12a^3}, \frac{10}{12a^3}$, respectively.

The multipliers $3a$, $4a^2$, and 2 are obtained by dividing $12a^3$, the L. C. M. of the denominators, by the respective denominators of the given fractions.

171. Therefore, to reduce fractions to equivalent fractions having the lowest common denominator,

Find the L. C. M. of the denominators.

Divide the L. C. M. by the denominator of each fraction.

Multiply the first numerator by the first quotient, the second by the second quotient, and so on.

The products will be the numerators of the equivalent fractions.

The L. C. M. of the given denominators will be the denominator of each of the equivalent fractions.

EXERCISE 55.

Reduce to equivalent fractions with the lowest common denominator:

1. $\frac{3x-7}{6}, \frac{4x-9}{18}$.
2. $\frac{2x-4y}{5x^2}, \frac{3x-8y}{10x}$.
3. $\frac{4a-5c}{5ac}, \frac{3a-2c}{12a^2c}$.
4. $\frac{5}{1-x}, \frac{6}{1-x^2}$.
5. $\frac{1}{(a-b)(b-c)}, \frac{1}{(a-b)(a-c)}$.
6. $\frac{4x^2}{3(a+b)}, \frac{xy}{6(a^2-b^2)}$.
7. $\frac{8x+2}{x-2}, \frac{2x-1}{3x-6}, \frac{3x+2}{5x-10}$.
8. $\frac{a-bm}{mx}, 1, \frac{c-bn}{nx}$.

ADDITION AND SUBTRACTION OF FRACTIONS.

172. To add fractions:

Reduce the fractions to equivalent fractions having the lowest common denominator.

Add the numerators of the equivalent fractions.

Write the result over the lowest common denominator.

173. To subtract one fraction from another:

Reduce the fractions to equivalent fractions having the lowest common denominator.

Subtract the numerator of the subtrahend from the numerator of the minuend.

Write the result over the lowest common denominator.

(1) Simplify, $\frac{4x+7}{5} + \frac{3x-4}{15}.$

The lowest common denominator (L. C. D.) = 15.

The multipliers are 3 and 1 respectively.

$$12x + 21 = \text{1st numerator,}$$

$$3x - 4 = \text{2d numerator,}$$

$$15x + 17 = \text{sum of numerators.}$$

$$\therefore \frac{4x+7}{5} + \frac{3x-4}{15} = \frac{15x+17}{15}.$$

(2) Simplify, $\frac{3a-4b}{7} - \frac{2a-b+c}{3} + \frac{13a-4c}{12}.$

The L. C. D. = 84.

The multipliers are 12, 28, and 7 respectively.

$$36a - 48b = \text{1st numerator,}$$

$$-56a + 28b - 28c = \text{2d numerator,}$$

$$91a - 28c = \text{3d numerator.}$$

$$71a - 20b - 56c = \text{sum of numerators.}$$

$$\therefore \frac{3a-4b}{7} - \frac{2a-b+c}{3} + \frac{13a-4c}{12} = \frac{71a-20b-56c}{84}.$$

NOTE. Since the *minus sign* precedes the second fraction, the signs of all the terms of the numerator of this fraction are changed after being multiplied by 28.

EXERCISE 56.

Simplify :

$$1. \frac{3x-2y}{5x} + \frac{5x-7y}{10x} + \frac{8x+2y}{25}.$$

$$2. \frac{4x^2-7y^2}{3x^2} + \frac{3x-8y}{6x} + \frac{5-2y}{12}.$$

$$3. \frac{4a^2+5b^2}{2b^2} + \frac{3a+2b}{5b} + \frac{7-2a}{9}.$$

$$4. \frac{4x+5}{3} - \frac{3x-7}{5x} + \frac{9}{12x^2}.$$

$$5. \frac{4x-3y}{7} + \frac{3x+7y}{14} - \frac{5x-2y}{21} + \frac{9x+2y}{42}.$$

$$6. \frac{3xy-4}{x^2y^2} - \frac{5y^2+7}{xy^3} - \frac{6x^2-11}{x^3y}.$$

$$7. \frac{a^2-2ac+c^2}{a^2c^2} - \frac{b^2-2bc+c^2}{b^2c^2}.$$

$$8. \frac{5a^3-2}{8a^2} - \frac{3a^2-a}{8}.$$

$$9. \frac{a-b}{c} + \frac{b-c}{a} + \frac{c-a}{b} + \frac{ab^2+bc^2+ca^2}{abc}.$$

$$10. \frac{1}{2x^2y} - \frac{1}{6y^2z} - \frac{1}{2xz^2} + \frac{2x-z}{4x^2z^2} + \frac{y-2z}{4x^2yz}.$$

Simplify $\frac{x-y}{x+y} + \frac{x+y}{x-y}$.

The L. C. D. = $x^2 - y^2$.

The multipliers are $x-y$ and $x+y$, respectively.

$$\frac{x^2 - 2xy + y^2}{2x^2} = \text{1st numerator.}$$

$$\frac{x^2 + 2xy + y^2}{2y^2} = \text{2d numerator.}$$

$$\frac{2x^2}{2x^2} + \frac{2y^2}{2y^2} = \text{sum of numerators}$$

$$\text{or, } 2(x^2 + y^2) = \text{“ “ “}$$

$$\therefore \frac{x-y}{x+y} + \frac{x+y}{x-y} = \frac{2(x^2 + y^2)}{x^2 - y^2}.$$

EXERCISE 57.

Simplify:

$$1. \frac{1}{x-6} + \frac{1}{x+5}.$$

$$6. \frac{1}{2a(a+x)} + \frac{1}{2a(a-x)}.$$

$$2. \frac{1}{x-7} - \frac{1}{x-3}.$$

$$7. \frac{a}{(a+b)b} - \frac{b}{(a-b)a}.$$

$$3. \frac{1}{1+x} + \frac{1}{1-x}.$$

$$8. \frac{5}{2x(x-1)} - \frac{3}{4x(x-2)}.$$

$$4. \frac{1}{1-x} - \frac{2}{1-x^2}.$$

$$9. \frac{1+x}{1+x+x^2} - \frac{1-x}{1-x+x^2}.$$

$$5. \frac{1}{x-y} + \frac{x}{(x-y)^2}.$$

$$10. \frac{2ax-3by}{2xy(x-y)} - \frac{2ax+3by}{2xy(x+y)}.$$

$$(1) \text{ Simplify } \frac{2a+b}{a-b} - \frac{2a-b}{a+b} - \frac{6ab}{a^2-b^2}.$$

The L. C. D. = $(a-b)(a+b)$.

The multipliers are $a+b$, $a-b$, and 1, respectively.

$$\begin{array}{rcl}
 2a^2 + 3ab + b^2 & = & \text{1st numerator,} \\
 -2a^2 + 3ab - b^2 & = & \text{2d numerator,} \\
 \hline
 -6ab & = & \text{3d numerator,} \\
 0 & = & \text{sum of numerators.}
 \end{array}$$

$$\therefore \frac{2a+b}{a-b} - \frac{2a-b}{a+b} - \frac{6ab}{a^2-b^2} = 0.$$

(2) Simplify $\frac{y^2}{x^2-y^2} - \frac{x-y}{x+y} + 1 + \frac{2xy}{x^2+y^2}$.

The L. C. D. = $(x+y)(x-y)(x^2+y^2)$.

The multipliers are, respectively:

x^2+y^2 , $(x-y)(x^2+y^2)$, $(x+y)(x-y)(x^2+y^2)$, $(x+y)(x-y)$.

$$\begin{array}{rcl}
 x^2y^2 & + & y^4 = \text{1st numerator,} \\
 -x^4 + 2x^3y - 2x^2y^2 + 2xy^3 - y^4 & = & \text{2d numerator,} \\
 x^4 & - & y^4 = \text{3d numerator,} \\
 \hline
 2x^3y & - & 2xy^3 = \text{4th numerator,} \\
 4x^3y - x^2y^2 & - & y^4 = \text{sum of numerators.}
 \end{array}$$

$$\therefore \text{Sum of fractions} = \frac{4x^3y - x^2y^2 - y^4}{x^4 - y^4}.$$

EXERCISE 58.

Simplify:

1. $\frac{1}{1+a} + \frac{1}{1-a} + \frac{2a}{1-a^2}$ 3. $\frac{x}{1-x} - \frac{x^2}{1-x} + \frac{x}{1+x^2}$

2. $\frac{1}{1-x} - \frac{1}{1+x} + \frac{2x}{1+x^2}$ 4. $\frac{x}{y} + \frac{y}{x+y} + \frac{x^2}{x^2+xy}$

5. $\frac{x-1}{x-2} + \frac{x-2}{x-3} + \frac{x-3}{x-4}$

6. $\frac{3}{x-a} + \frac{4a}{(x-a)^2} - \frac{5a^2}{(x-a)^3}$

7. $\frac{1}{x-1} - \frac{1}{x+2} - \frac{3}{(x+1)(x+2)}$
8. $\frac{a-b}{(b+c)(c+a)} + \frac{b-c}{(c+a)(a+b)} + \frac{c-a}{(a+b)(b+c)}$
9. $\frac{x-a}{x-b} + \frac{x-b}{x-a} - \frac{(a-b)^2}{(x-a)(x-b)}$
10. $\frac{x+y}{y} - \frac{2x}{x+y} + \frac{x^2y-x^2}{y(x^2-y^2)}$
11. $\frac{a+b}{(b-c)(c-a)} + \frac{b+c}{(c-a)(a-b)} + \frac{c+a}{(a-b)(b-c)}$
12. $\frac{a^2-bc}{(a+b)(a+c)} + \frac{b^2-ac}{(b+a)(b+c)} + \frac{c^2+ab}{(c+b)(c+a)}$
13. $\frac{a}{a-x} - \frac{x}{a+2x} - \frac{a^2+x^2}{(a-x)(a+2x)}$
14. $\frac{3}{(a-b)(b-c)} - \frac{4}{(a-b)(a-c)} + \frac{6}{(a-c)(b-c)}$
15. $\frac{x-2y}{x(x-y)} - \frac{2x+y}{y(x+y)} - \frac{2x}{x^2-y^2}$
16. $\frac{a-b}{x(a+b)} - \frac{a-b}{y(a+b)} - \frac{(a-b)(x+y)}{xy(a+b)}$
17. $\frac{3x}{(x+y)^2} - \frac{x+2y}{x^2-y^2} + \frac{3y}{(x-y)^2}$
18. $\frac{a-c}{(a+b)^2-c^2} - \frac{a-b}{(a+c)^2-b^2}$
19. $\frac{a+b}{ax+by} - \frac{a-b}{ax-by} + \frac{ab(x-y)}{a^2x^2-b^2y^2}$

174. Since $\frac{ab}{b} = a$, and $\frac{-ab}{-b} = a$,

it is evident that if the signs of both numerator and denominator are changed, the value of the fraction is not altered.

Again, $\frac{a-b}{c-d} = \frac{-(a-b)}{-(c-d)} = \frac{-a+b}{-c+d} = \frac{b-a}{d-c}.$

Hence, if the numerator or denominator be a compound expression, or if both be compound expressions, the sign of every term in the denominator may be changed, provided the sign of every term in the numerator be changed.

Since the change of the sign before the fraction is equivalent to the change of the sign before every term of the numerator of the fraction, *the sign before every term of the denominator may be changed, provided the sign before the fraction be changed.*

Since, also, the product of $+a$ multiplied by $+b$ is ab , and the product of $-a$ multiplied by $-b$ is ab , the signs of *two factors*, or of *any even number of factors*, of the denominator of a fraction may be changed without altering the value of the fraction.

By the application of these principles, fractions may often be changed to a more simple form for addition or subtraction.

(1) Simplify $\frac{2}{x} - \frac{3}{2x-1} + \frac{2x-3}{1-4x^2}.$

Change the signs before the terms of the denominator of the third fraction, and change the sign before the fraction.

The result is,

$$\frac{2}{x} - \frac{3}{2x-1} - \frac{2x-3}{4x^2-1}.$$

in which the several denominators are *arranged in the same order with respect to x.*

The L. C. D. = $x(2x-1)(2x+1)$.

$$\begin{array}{rcl} 8x^3 - 2 & = & \text{1st numerator,} \\ -6x^3 - 3x & = & \text{2d numerator,} \\ -2x^3 + 3x & = & \text{3d numerator.} \\ \hline -2 & = & \text{sum of numerators.} \end{array}$$

$$\therefore \text{Sum of the fractions} = \frac{-2}{x(2x-1)(2x+1)}.$$

(2) Simplify :

$$\frac{1}{a(a-b)(a-c)} + \frac{1}{b(b-a)(b-c)} + \frac{1}{c(c-a)(c-b)}.$$

Change the sign of the factor $(b-a)$ in the denominator of the second fraction, and change the sign before the fraction.

Then change the signs of the factors $(c-a)$ and $(c-b)$ in the denominator of the third fraction.

The result is,

$$\frac{1}{a(a-b)(a-c)} - \frac{1}{b(a-b)(b-c)} + \frac{1}{c(a-c)(b-c)}.$$

in which the factors of the several denominators are written in alphabetical order.

The L. C. D. = $abc(a-b)(a-c)(b-c)$.

$$\begin{array}{rcl} bc(b-c) & = & b^2c - bc^2 \quad = \text{1st numerator,} \\ -ac(a-c) & = & -a^2c + ac^2 \quad = \text{2d numerator,} \\ ab(a-b) & = & a^2b - ab^2 \quad = \text{3d numerator.} \\ \hline a^2b - a^2c - ab^2 + ac^2 + b^2c - bc^2 & = & \text{sum of numerators.} \end{array}$$

$$\begin{aligned} &= a^2(b-c) - a(b^2-c^2) + bc(b-c), \\ &= [a^2 - a(b+c) + bc][b-c], \\ &= [a^2 - ab - ac + bc][b-c], \\ &= [(a^2 - ac) - (ab - bc)][b-c], \\ &= [a(a-c) - b(a-c)][b-c], \\ &= (a-b)(a-c)(b-c). \end{aligned}$$

$$\begin{aligned}\therefore \text{Sum of the fractions} &= \frac{(a-b)(a-c)(b-c)}{abc(a-b)(a-c)(b-c)} \\ &= \frac{1}{abc}.\end{aligned}$$

EXERCISE 59.

Simplify :

1. $\frac{x}{x-y} + \frac{x-y}{y-x}.$
2. $\frac{3+2x}{2-x} + \frac{3x-2}{2+x} + \frac{16x-x^2}{x^2-4}.$
3. $\frac{x^2}{x^2-1} + \frac{x}{x+1} - \frac{x}{1-x}.$
4. $\frac{4}{3-3y^2} + \frac{1}{2-2y} + \frac{1}{6y+6}.$
5. $\frac{1}{(2-m)(3-m)} - \frac{2}{(m-1)(m-3)} + \frac{1}{(m-1)(m-2)}.$
6. $\frac{1}{(b-a)(x+a)} + \frac{1}{(a-b)(x+b)}.$
7. $\frac{a^2+b^2}{a^2-b^2} + \frac{2ab^2}{b^2-a^2} + \frac{2a^2b}{a^2+b^2}.$
8. $\frac{b-a}{x-b} - \frac{a-2b}{b+x} - \frac{3x(a-b)}{b^2-x^2}.$
9. $\frac{3+2x}{2-x} - \frac{2-3x}{2+x} + \frac{16x-x^2}{x^2-4}.$
10. $\frac{3}{1-2x} - \frac{7}{1+2x} - \frac{4-20x}{4x^2-1}.$

$$11. \frac{a+b}{(b-c)(c-a)} + \frac{b+c}{(b-a)(a-c)} + \frac{c+a}{(a-b)(b-c)}.$$

$$12. \frac{a^2-bc}{(a-b)(a-c)} + \frac{b^2+ac}{(b+c)(b-a)} + \frac{c^2+ab}{(c-a)(c+b)}.$$

$$13. \frac{y+z}{(x-y)(x-z)} + \frac{z+x}{(y-x)(y-z)} + \frac{x+y}{(z-x)(z-y)}.$$

$$14. \frac{3}{(a-b)(b-c)} - \frac{4}{(b-a)(c-a)} - \frac{6}{(a-c)(c-b)}.$$

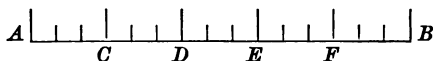
$$15. \frac{1}{x(x-y)(x-z)} + \frac{1}{y(y-x)(y-z)} - \frac{1}{xyz}.$$

MULTIPLICATION OF FRACTIONS.

175. In Arithmetic it is often necessary to take equal parts of *fractions of units*.

Suppose it is required to take $\frac{2}{3}$ of $\frac{1}{5}$ of a unit.

Let the line AB represent the unit of length.



Suppose AB divided into 5 equal parts, at C , D , E , and F , and each of these parts to be subdivided into 3 equal subdivisions.

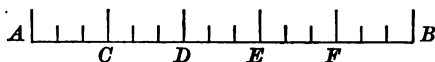
Then one of the parts, as AC , will contain 3 of these subdivisions, and the whole line AB will contain 15 of these subdivisions.

That is, $\frac{1}{5}$ of $\frac{1}{5}$ of the line will be $\frac{1}{15}$ of the line;

$\frac{1}{3}$ of $\frac{1}{5}$ will be $\frac{1}{15} + \frac{1}{15} + \frac{1}{15}$, or $\frac{3}{15}$ of the line; and

$\frac{2}{3}$ of $\frac{1}{5}$ will be *twice* $\frac{3}{15}$, or $\frac{6}{15}$ of the line.

Suppose it is required to take $\frac{c}{d}$ of $\frac{a}{b}$ of the line AB , where a, b, c, d represent positive whole numbers.



Let the line AB be divided into b equal parts, and let each of these parts be subdivided into d equal subdivisions.

Then the whole line will contain bd of these subdivisions, and one of these subdivisions will be $\frac{1}{bd}$ of the line.

If one of the subdivisions be taken from each of a parts, they will together be $\frac{a}{bd}$ of the line. That is,

$$\frac{1}{d} \text{ of } \frac{a}{b} = \frac{1}{bd} + \frac{1}{bd} + \frac{1}{bd} \dots \text{ taken } a \text{ times,} = \frac{a}{bd},$$

$$\text{and } \frac{c}{d} \text{ of } \frac{a}{b} \text{ will be } c \text{ times } \frac{a}{bd}, \text{ or } \frac{ac}{bd} \text{ of the line.}$$

Therefore, to find a fraction of a fraction,

Find the product of the numerators for the numerator of the product, and of the denominators for the denominator of the product.

176. Now, $\frac{c}{d} \times \frac{a}{b}$ means $\frac{c}{d}$ of $\frac{a}{b}$.

Therefore, to find the product of two fractions,

Find the product of the numerators for the numerator of the product, and of the denominators for the denominator of the product.

The same rule will hold whatever be the values of a, b, c, d .

If a factor exist in both a numerator and a denominator it may be cancelled; for the cancelling of a common factor *before* the multiplication is evidently equivalent to cancelling it *after* the multiplication; and this may be done by § 163.

DIVISION OF FRACTIONS.

177. Multiplying by the reciprocal of a number is equivalent to dividing by the number. Thus, multiplying by $\frac{1}{4}$ is equivalent to dividing by 4.

The reciprocal of a fraction is the fraction with its terms interchanged.

Thus, the reciprocal of $\frac{3}{4}$ is $\frac{4}{3}$, for $\frac{3}{4} \times \frac{4}{3} = 1$. § 42.

Therefore, to divide by a fraction,

Interchange the terms of the fraction, and multiply by the resulting fraction. Thus,

$$(1) \quad \frac{2a}{3x^2} \div \frac{1}{3x} = \frac{3x}{1} \times \frac{2a}{3x^2} = \frac{2a}{x}.$$

The common factor cancelled is $3x$.

$$(2) \quad \frac{14x^2}{27y^2} \div \frac{7x}{9y} = \frac{9y}{7x} \times \frac{14x^2}{27y^2} = \frac{2x}{3y}.$$

The common factors cancelled are $9y$ and $7x$.

$$(3) \quad \frac{ax}{(a-x)^2} \div \frac{ab}{a^2-x^2} = \frac{(a+x)(a-x)}{ab} \times \frac{ax}{(a-x)(a-x)} \\ = \frac{x(a+x)}{b(a-x)}.$$

The common factors cancelled are a and $a-x$.

If the divisor be an integral expression, it may be changed to the fractional form. § 162.

EXERCISE 60.

Simplify :

$$1. \quad \frac{a}{bx} \times \frac{cx}{d}.$$

$$3. \quad \frac{3p}{2p-2} \div \frac{2p}{p-1}.$$

$$2. \quad \frac{2x}{a} \times \frac{3ab}{c} \times \frac{3ac}{2b}.$$

$$4. \quad \frac{8x^4y}{15ab^2} \div \frac{2x^3}{3ab^2}.$$

$$5. \frac{8a^2b^3}{45x^2y} \times \frac{15xy^3}{24a^3b^2}.$$

$$8. \frac{9m^2n^3}{8p^3q^3} \times \frac{5p^2q}{2xy} \times \frac{24x^2y^3}{90mn}.$$

$$6. \frac{9x^2y^2z}{10a^2b^2c} \times -\frac{20a^3b^3c}{18xy^2z}.$$

$$9. \frac{25k^3n^2}{14n^3q^3} \times \frac{70n^3q}{75p^2m} \times \frac{3pm}{4k^2n}.$$

$$7. \frac{3x^2y}{4xz^2} \times \frac{5y^2z}{6xy} \times -\frac{12x^3}{2xy^3}.$$

$$10. \frac{a-b}{a^2+ab} \times \frac{a^3-b^3}{a^3-ab}.$$

$$11. \frac{a^2+b^2}{a^2-b^2} \div \frac{a-b}{a+b}.$$

$$12. \frac{x^3+x-2}{x^3-7x} \times \frac{x^3-13x+42}{x^3+2x}.$$

$$13. \frac{x^2-11x+30}{x^2-6x+9} \times \frac{x^2-3x}{x^2-5x}.$$

$$14. \frac{a^3-x^3}{a^3+x^3} \times \frac{(a+x)^3}{(a-x)^3}.$$

$$15. \frac{2a(x^3-y^3)^3}{cx} \times \frac{x^3}{(x-y)(x+y)^3}.$$

$$16. \frac{a^3+2ab}{a^2+4b^2} \times \frac{ab-2b^3}{a^2-4b^2}.$$

$$18. \frac{x^3+xy}{x-y} \times \frac{(x-y)^3}{x^4-y^4}.$$

$$17. \frac{x^2-4}{x^2+5x} \times \frac{x^2-25}{x^2+2x}.$$

$$19. \frac{m^3-n^3}{c^3+d^3} \div \frac{n-m}{c+d}.$$

$$20. \frac{a^2-4a+3}{a^2-5a+4} \times \frac{a^2-9a+20}{a^2-10a+21} \times \frac{a^2-7a}{a^2-5a}.$$

$$21. \frac{b^2-7b+6}{b^2+3b-4} \times \frac{b^2+10b+24}{b^2-14b+48} \div \frac{b^2+6b}{b^3-8b^2}.$$

$$22. \frac{x^2-y^2}{x^2-3xy+2y^2} \times \frac{xy-2y^2}{x^2+xy} \times \frac{x^2-xy}{(x-y)^2}.$$

$$23. \frac{a^3-3a^2b+3ab^2-b^3}{a^2-b^2} \div \frac{2ab-2b^2}{3} \times \frac{a^2+ab}{a-b}.$$

$$24. \frac{(a+b)^2 - c^2}{a^2 - (b-c)^2} \div \frac{c^2 - (a+b)^2}{c^2 - (a-b)^2}.$$

$$25. \frac{(x-a)^2 - b^2}{(x-b)^2 - a^2} \times \frac{x^2 - (b-a)^2}{x^2 - (a-b)^2}.$$

$$26. \frac{(a+b)^2 - (c+d)^2}{(a+c)^2 - (b+d)^2} \div \frac{(a-c)^2 - (d-b)^2}{(a-b)^2 - (d-c)^2}.$$

$$27. \frac{x^2 - 2xy + y^2 - z^2}{x^2 + 2xy + y^2 - z^2} \times \frac{x + y - z}{x - y + z}.$$

COMPLEX FRACTIONS.

178. A **complex fraction** is one which has a fraction in the numerator or in the denominator, or in both.

To simplify a complex fraction,

Divide the numerator by the denominator.

(1) Simplify $\frac{\frac{1}{2}}{\frac{1}{2}}$.

$$\frac{\frac{1}{2}}{\frac{1}{2}} = \frac{1}{2} \div \frac{1}{2} = \frac{1}{1} \times \frac{2}{2} = \frac{2}{2}.$$

(2) Simplify $\frac{2\frac{2}{3}}{5\frac{1}{3}}$.

$$\frac{2\frac{2}{3}}{5\frac{1}{3}} = \frac{\frac{8}{3}}{\frac{16}{3}} = \frac{8}{3} \div \frac{16}{3} = \frac{8}{3} \times \frac{3}{16} = \frac{64}{144}.$$

(3) Simplify $\frac{3x}{x - \frac{1}{4}}$.

$$\begin{aligned} \frac{3x}{x - \frac{1}{4}} &= \frac{3x}{\frac{4x-1}{4}} = \frac{3x}{1} \div \frac{4x-1}{4} = \frac{4}{4x-1} \times \frac{3x}{1} \\ &= \frac{12x}{4x-1}. \end{aligned}$$

179. It is often shorter to multiply both terms of the fraction by the L. C. D. of the fractions contained in the numerator and denominator.

Thus, in § 178, (1), multiply both terms by 6; (2), both terms by 24; (3), both terms by 4. The results obtained are $\frac{2}{3}$, $\frac{64}{141}$, $\frac{12x}{4x-1}$, respectively.

Simplify by first simplifying the denominator :

$$\begin{aligned} & \frac{x}{1 - \frac{x}{1+x + \frac{x}{1-x+x^2}}} \\ & \frac{x}{1 - \frac{x}{1+x + \frac{x}{1-x+x^2}}} = \frac{x}{1 - \frac{x(1-x+x^2)}{(1+x)(1-x+x^2)+x}} \\ & = \frac{x}{1 - \frac{x-x^2+x^3}{1+x+x^3}} \\ & = \frac{x(1+x+x^3)}{1+x+x^3-(x-x^2+x^3)} \\ & = \frac{x+x^2+x^4}{1+x^3}. \end{aligned}$$

The expression $\frac{x}{1+x + \frac{x}{1-x+x^2}}$ is reduced to the form

$$\frac{x(1-x+x^2)}{(1+x)(1-x+x^2)+x}, \text{ and this is equal to } \frac{x-x^2+x^3}{1+x+x^3}.$$

The expression $\frac{x}{1 - \frac{x-x^2+x^3}{1+x+x^3}}$ is reduced to the form

$$\frac{x(1+x+x^3)}{1+x+x^3-(x-x^2+x^3)}, \text{ and this is equal to } \frac{x+x^2+x^4}{1+x^3}.$$

EXERCISE 61.

Simplify :

$$1. \frac{\frac{3x}{2} + \frac{x-1}{3}}{\frac{13}{6}(x+1) - \frac{x}{3} - 2\frac{1}{2}}$$

$$8. 1 - \frac{1}{1 + \frac{1}{x}}$$

$$2. \frac{x-1 + \frac{6}{x-6}}{x-2 + \frac{3}{x-6}}$$

$$9. 1 + \frac{x}{1+x + \frac{2x^2}{1-x}}$$

$$3. \frac{3}{x+1} - \frac{2x-1}{x^2 + \frac{x}{2} - \frac{1}{2}}$$

$$10. \frac{1}{1 - \frac{1}{1 + \frac{1}{x}}}$$

$$4. \frac{x-a}{x - \frac{(x-b)(x-c)}{x+a}}$$

$$11. \frac{1}{1 + \frac{x}{1+x + \frac{2x^2}{1-x}}}$$

$$5. \frac{\left(\frac{a}{x} - \frac{x}{a}\right)\left(\frac{a}{x} + \frac{x}{a}\right)}{1 - \frac{x-a}{x+a}}$$

$$12. \frac{\left(\frac{a}{x} + \frac{x}{a} - 2\right)\left(\frac{a}{x} + \frac{x}{a} + 2\right)}{\left(\frac{a}{x} - \frac{x}{a}\right)^2}$$

$$6. \frac{\frac{1}{x-y} - \frac{x}{x^2-y^2}}{\frac{x}{xy+y^2} - \frac{y}{x^2+xy}}$$

$$13. \frac{\frac{x^2+y^2}{x^2-y^2} + \frac{2x}{x+y} \left\{ \frac{xy-x^2}{(x-y)^3} + \frac{x+y}{x-y} \right\}}{x-y}$$

$$7. \frac{\frac{x+1}{x-1} + \frac{x-1}{x+1}}{\frac{x+1}{x-1} - \frac{x-1}{x+1}}$$

$$14. \frac{\frac{(x^2-y^2)(2x^2-2xy)}{4(x-y)^2}}{\frac{xy}{x+y}}$$

$$15. \frac{\frac{ab}{x^2 + (a+b)x + ab} - \frac{ac}{x^2 + (a+c)x + ac}}{\frac{b-c}{x^2 + (b+c)x + bc}}$$

$$16. \frac{x}{1 + \frac{1}{x}} + 1 - \frac{1}{x+1}$$

$$17. \frac{\frac{a+b}{b} + \frac{b}{a+b}}{\frac{1}{a} + \frac{1}{b}}$$

$$18. \frac{2m-3 + \frac{1}{m}}{\frac{2m-1}{m}}$$

$$19. \frac{\frac{1}{ab} + \frac{1}{ac} + \frac{1}{bc}}{\frac{a^3 - (b+c)^3}{ab}}$$

$$20. \frac{3}{1 + \frac{3}{1 + \frac{3}{1-x}}}$$

EXERCISE 62.

MISCELLANEOUS EXAMPLES.

$$1. \text{ Simplify } \frac{x^4 - 9x^3 + 7x^2 + 9x - 8}{x^4 + 7x^3 - 9x^2 - 7x + 8}.$$

$$2. \text{ Find the value of } \frac{a^2 + b^2 - c^2 + 2ab}{a^2 - b^2 - c^2 + 2bc} \text{ when } a=4, b=\frac{1}{2}, c=1.$$

$$3. \text{ Find the value of } 3a^2 + \frac{2ab^2}{c} - \frac{c^3}{b^2} \text{ when } a=4, b=\frac{1}{2}, c=1.$$

$$4. \text{ Simplify } \frac{2}{(x^2-1)^2} - \frac{1}{2x^2-4x+2} - \frac{1}{1-x^2}.$$

$$5. \text{ Simplify } \left(\frac{x}{1 + \frac{1}{x}} + 1 - \frac{1}{x+1} \right) \div \left(\frac{x}{1 - \frac{1}{x}} - x - \frac{1}{x-1} \right).$$

$$6. \text{ Find the value of } \left(\frac{x-a}{x-b} \right)^3 - \frac{x-2a+b}{x+a-2b} \text{ when } x = \frac{a+b}{2}.$$

$$7. \text{ Simplify } \left\{ \frac{a+b}{2(a-b)} - \frac{a-b}{2(a+b)} + \frac{2b^2}{a^2-b^2} \right\} \frac{a-b}{2b}.$$

8. Simplify $\left(\frac{x^2+y^2}{x^2-y^2} - \frac{x^2-y^2}{x^2+y^2}\right) \div \left(\frac{x+y}{x-y} - \frac{x-y}{x+y}\right)$.

9. Simplify

$$\left(\frac{x^2}{y^2} - 1\right)\left(\frac{x}{x-y} - 1\right) + \left(\frac{x^2}{y^2} - 1\right)\left(\frac{x^2+xy}{x^2+xy+y^2} - 1\right).$$

10. Simplify

$$\left(\frac{a^2-ab}{a^3-b^3}\right)\left(\frac{a^2+ab+b^2}{a+b}\right) + \left(\frac{2a^3}{a^3+b^3} - 1\right)\left(1 - \frac{2ab}{a^2+ab+b^2}\right).$$

11. Simplify $\frac{1 + \frac{a-x}{a+x}}{1 - \frac{a-x}{a+x}} \div \frac{1 + \frac{a^2-x^2}{a^2+x^2}}{1 - \frac{a^2-x^2}{a^2+x^2}}$.

12. Divide $x^3 + \frac{1}{x^3} - 3\left(\frac{1}{x^2} - x^2\right) + 4\left(x + \frac{1}{x}\right)$ by $x + \frac{1}{x}$.

13. Simplify $\frac{1 - \frac{2xy}{(x+y)^2}}{1 + \frac{2xy}{(x-y)^2}} \div \left\{ \frac{1 - \frac{y}{x}}{1 + \frac{y}{x}} \right\}^2$.

14. Find the value of $\frac{x+2a}{2b-x} + \frac{x-2a}{2b+x} - \frac{4ab}{4b^2-x^2}$ when $x = \frac{ab}{a+b}$.

15. Find the value of $\frac{x+y-1}{x-y+1}$ when $x = \frac{a+1}{ab+1}$ and $y = \frac{ab+a}{ab+1}$.

16. Simplify

$$\frac{1}{a(a-b)(a-c)} + \frac{1}{b(b-c)(b-a)} + \frac{1}{c(c-a)(c-b)}.$$

17. Simplify $\frac{3abc}{bc+ca-ab} - \frac{\frac{a-1}{a} + \frac{b-1}{b} + \frac{c-1}{c}}{\frac{1}{a} + \frac{1}{b} - \frac{1}{c}}.$

$$18. \text{ Simplify } \frac{\frac{m^2+n^2}{n} - m}{\frac{1}{n} - \frac{1}{m}} \times \frac{m^2-n^2}{m^3+n^3}$$

$$19. \text{ Simplify } \frac{\frac{1}{a} + \frac{1}{b+c}}{\frac{1}{a} - \frac{1}{b+c}} \left\{ 1 + \frac{b^2+c^2-a^2}{2bc} \right\}$$

$$20. \text{ Simplify } 3a - [b + \{2a - (b-c)\}] + \frac{1}{2} + \frac{2c^2-1}{2c+1}$$

$$21. \text{ Simplify } \frac{\frac{1}{a-x} - \frac{1}{a-y} + \frac{x}{(a-x)^2} - \frac{y}{(a-y)^2}}{\frac{1}{(a-y)(a-x)^2} - \frac{1}{(a-x)(a-y)^2}}$$

$$22. \text{ Simplify } \frac{1}{x + \frac{1}{1 + \frac{x+1}{3-x}}}$$

$$23. \frac{(x^2-y^2)(2x^2-2xy)}{4(x-y)^2 \div \frac{xy}{x+y}}$$

$$24. \text{ Simplify } \left(\frac{c-b}{c+b} - \frac{c^3-b^3}{c^3+b^3} \right) \div \left(\frac{c+b}{c-b} + \frac{c^2+b^2}{c^2-b^2} \right)$$

$$25. \text{ Simplify } \frac{y}{(x-y)(x-z)} + \frac{x}{(y-x)(y-z)} + \frac{x+y}{(z-x)(z-y)}$$

$$26. \text{ Simplify } \frac{1}{a(a-b)(a-c)} + \frac{1}{b(b-a)(b-c)} - \frac{1}{abc}$$

$$27. \text{ Simplify } \frac{x-4 + \frac{6}{x+1}}{x - \frac{6}{x-1}} \times \frac{1 - \frac{x+5}{x^2-1}}{(x-1)(x-2)}$$

CHAPTER IX.

FRACTIONAL EQUATIONS.

TO REDUCE EQUATIONS CONTAINING FRACTIONS.

180. (1) $\frac{x}{2} + \frac{x}{4} = 12.$

Multiply both sides by 4, the L. C. M. of the denominators.

$$\begin{aligned}\text{Then,} \quad 2x + x &= 48, \\ 3x &= 48. \\ \therefore x &= 16.\end{aligned}$$

(2) $\frac{x}{6} - 4 = 24 - \frac{x}{8}.$

Multiply both sides by 24, the L. C. M. of the denominators.

$$\begin{aligned}\text{Then,} \quad 4x - 96 &= 576 - 3x, \\ 4x + 3x &= 576 + 96, \\ 7x &= 672. \\ \therefore x &= 96.\end{aligned}$$

(3) $\frac{x}{3} - \frac{x-1}{11} = x - 9.$

Multiply by 33, the L. C. M. of the denominators.

$$\begin{aligned}\text{Then,} \quad 11x - 3x + 3 &= 33x - 297, \\ 11x - 3x - 33x &= -297 - 3, \\ -25x &= -300. \\ \therefore x &= 12.\end{aligned}$$

Since the minus sign precedes the second fraction, in removing the denominator, the + (understood) before x , the first term of the numerator, is changed to $-$, and the $-$ before 1, the second term of the numerator, is changed to $+$.

181. Therefore, to clear an equation of fractions,
Multiply each term by the L. C. M. of the denominators.

If a fraction is preceded by a **minus sign**, the *sign of every term of the numerator must be changed when the denominator is removed.*

EXERCISE 63.

Solve the equations :

$$1. \ 5x - \frac{x+2}{2} = 71.$$

$$4. \ \frac{5x}{2} - \frac{5x}{4} = \frac{9}{4} - \frac{3-x}{2}.$$

$$2. \ x - \frac{3-x}{3} = \frac{17}{3}$$

$$5. \ 2x - \frac{5x-4}{6} = 7 - \frac{1-2x}{5}.$$

$$3. \ \frac{5-2x}{4} + 2 = x - \frac{6x-8}{2}. \quad 6. \ \frac{x+2}{2} = \frac{14}{9} - \frac{3+5x}{4}.$$

$$7. \ \frac{5x+3}{8} - \frac{3-4x}{3} + \frac{x}{2} = \frac{31}{2} - \frac{9-5x}{6}.$$

$$8. \ \frac{10x+3}{3} - \frac{6x-7}{2} = 10(x-1).$$

$$9. \ \frac{5x-7}{2} - \frac{2x+7}{3} = 3x-14.$$

$$10. \ \frac{7x+5}{6} - \frac{5x-6}{4} = \frac{8-5x}{12}.$$

$$11. \ \frac{x+4}{3} - \frac{x-4}{5} = 2 + \frac{3x-1}{15}.$$

$$12. \ \frac{3x+5}{7} - \frac{2x+7}{3} + 10 - \frac{3x}{5} = 0.$$

$$13. \ \frac{1}{7}(3x-4) + \frac{1}{3}(5x+3) = 43-5x.$$

$$14. \ \frac{1}{2}(27-2x) = \frac{9}{2} - \frac{1}{10}(7x-54).$$

$$15. 5x - \{8x - 3[16 - 6x - (4 - 5x)]\} = 6.$$

$$16. \frac{5x-3}{7} - \frac{9-x}{3} = \frac{5x}{2} + \frac{19}{6}(x-4).$$

$$17. \frac{2x+7}{7} - \frac{9x-8}{11} = \frac{x-11}{2}.$$

$$18. \frac{8x-15}{3} - \frac{11x-1}{7} = \frac{7x+2}{13}.$$

$$19. \frac{7x+9}{8} - \frac{3x+1}{7} = \frac{9x-13}{4} - \frac{249-9x}{14}.$$

182. If the denominators contain both simple and compound expressions, it is best to remove the simple expressions first, and then each compound expression in turn. After each multiplication the result should be reduced to the simplest form.

$$(1) \frac{8x+5}{14} + \frac{7x-3}{6x+2} = \frac{4x+6}{7}.$$

Multiply both sides by 14.

$$\text{Then,} \quad 8x+5 + \frac{49x-21}{3x+1} = 8x+12.$$

$$\text{Transpose and combine,} \quad \frac{49x-21}{3x+1} = 7.$$

$$\begin{aligned} \text{Multiply by } 3x+1, \quad 49x-21 &= 21x+7, \\ 28x &= 28. \\ \therefore x &= 1. \end{aligned}$$

$$(2) \frac{3-\frac{4x}{9}}{4} = \frac{1}{4} - \frac{\frac{7x}{9}-3}{10}.$$

Simplify the complex fractions by multiplying both terms of each fraction by 9.

$$\text{Then,} \quad \frac{27-4x}{36} = \frac{1}{4} - \frac{7x-27}{90}.$$

Multiply both sides by 180.

$$\begin{aligned} 135-20x &= 45-14x+54, \\ -6x &= -36. \\ \therefore x &= 6. \end{aligned}$$

EXERCISE 64.

Solve the equations:

1. $\frac{9x+20}{36} = \frac{4(x-3)}{5x-4} + \frac{x}{4}.$
2. $\frac{9(2x-3)}{14} + \frac{11x-1}{3x+1} = \frac{9x+11}{7}.$
3. $\frac{10x+17}{18} - \frac{12x+2}{13x-16} = \frac{5x-4}{9}.$
4. $\frac{6x+13}{15} - \frac{3x+5}{5x-25} = \frac{2x}{5}.$
5. $\frac{18x-22}{39-6x} + 2x + \frac{1+16x}{24} = 4\frac{1}{2} - \frac{101-64x}{24}.$
6. $\frac{6-5x}{15} - \frac{7-2x^2}{14(x-1)} = \frac{1+3x}{21} - \frac{10x-11}{30} + \frac{1}{105}.$
7. $\frac{9x+5}{14} + \frac{8x-7}{6x+2} = \frac{36x+15}{56} + \frac{41}{56}.$
8. $\frac{6x+7}{15} - \frac{2x-2}{7x-6} = \frac{2x+1}{5}.$
9. $\frac{6x+1}{15} - \frac{2x-4}{7x-16} = \frac{2x-1}{5}.$
10. $\frac{7x-6}{35} - \frac{x-5}{6x-101} = \frac{x}{5}.$

183. Literal equations are equations in which some or all of the *known* numbers are represented by letters; the numbers regarded as known numbers are usually represented by the *first* letters of the alphabet.

$$(1) (a-x)(a+x) = 2a^2 + 2ax - x^2.$$

$$\text{Then, } a^2 - x^2 = 2a^2 + 2ax - x^2, \\ -2ax = a^2.$$

$$\therefore x = -\frac{a}{2}.$$

$$(2) (x-a)(x-b) - (x-b)(x-c) = 2(x-a)(a-c).$$

$$(x^2 - ax - bx + ab) - (x^2 - bx - cx + bc) = 2(ax - cx - a^2 + ac), \\ x^2 - ax - bx + ab - x^2 + bx + cx - bc = 2ax - 2cx - 2a^2 + 2ac.$$

$$\text{That is, } -3ax + 3cx = -2a^2 + 2ac - ab + bc, \\ -3(a-c)x = -2a(a-c) - b(a-c), \\ -3x = -2a - b.$$

$$\therefore x = \frac{2a+b}{3}.$$

EXERCISE 65.

Solve the equations:

$$1. ax + bc = bx + ac. \quad 2. 2a - cx = 3c - 5bx.$$

$$3. a^2x + bx - c = b^2x + cx - d.$$

$$4. -ac^2 + b^2c + abcx = abc + cmx - ac^2x + b^2c - mc.$$

$$5. (a+x+b)(a+b-x) = (a+x)(b-x) - ab.$$

$$6. (a^2+x)^2 = x^2 + 4a^2 + a^4.$$

$$7. (a^2-x)(a^2+x) = a^4 + 2ax - x^2.$$

$$8. \frac{ax-b}{c} + a = \frac{x+ac}{c}.$$

$$10. ax - \frac{3a-bx}{2} = \frac{1}{2}.$$

$$9. \frac{a(b^2x+x^3)}{bx} = acx + \frac{ax^2}{b}.$$

$$11. 6a - \frac{4ax-2b}{3} = x.$$

$$12. \frac{x^2-a}{bx} - \frac{a-x}{b} = \frac{2x}{b} - \frac{a}{x}.$$

$$13. \frac{3}{c} - \frac{ab-x^2}{bx} = \frac{4x-ac}{cx}.$$

$$14. am - b - \frac{ax}{b} + \frac{x}{m} = 0.$$

$$15. \frac{3ax-2b}{3b} - \frac{ax-a}{2b} = \frac{ax}{b} - \frac{2}{3}.$$

$$16. \frac{ab+x}{b^2} - \frac{b^2-x}{a^2b} = \frac{x-b}{a^2} - \frac{ab-x}{b^2}.$$

$$17. ax - \frac{bx+1}{x} = \frac{a(x^2-1)}{x} \quad 19. \frac{ab}{x} = bc + d + \frac{1}{x}.$$

$$18. \frac{ax^2}{b-cx} + a + \frac{ax}{c} = 0. \quad 20. \frac{a(d^2+x^2)}{dx} = ac + \frac{ax}{d}.$$

EXERCISE 66.

Solve the equations:

$$1. \frac{x-3}{4(x-1)} = \frac{x-5}{6(x-1)} + \frac{1}{9}.$$

$$2. x + \frac{x}{x-1} = \frac{(x-2)(x+4)}{x+1}.$$

$$3. \frac{7}{x-1} = \frac{6x+1}{x+1} - \frac{3(1+2x^2)}{x^2-1}.$$

$$4. \frac{1}{2(x-3)} - \frac{1}{3(x-2)} = \frac{x-1}{(x-2)(x-3)}.$$

$$5. 1 - \frac{2(2x+3)}{9(7-x)} = \frac{6}{7-x} - \frac{5x+1}{4(7-x)}.$$

$$6. \frac{17}{x+3} - 4 = \frac{5(21+2x)}{3x+9} - 10.$$

$$7. \frac{x-7}{x+7} = \frac{2x-15}{2x-6} - \frac{1}{2(x+7)}.$$

$$8. \frac{x+4}{3x+5} + 1\frac{1}{2} = \frac{3x+8}{2x+3}.$$

$$9. \frac{132x+1}{3x+1} + \frac{8x+5}{x-1} = 52. \quad 11. \frac{3x-1}{2x-1} - \frac{4x-2}{3x-2} = \frac{1}{6}$$

$$10. \frac{2}{2x-3} + \frac{1}{x-2} = \frac{6}{3x+2}. \quad 12. \frac{3}{x-1} - \frac{x+1}{x-1} = \frac{x^2}{1-x^2}$$

$$13. \frac{x-4}{x-5} - \frac{x-5}{x-6} = \frac{x-7}{x-8} - \frac{x-8}{x-9}$$

$$14. (x-a)(x-b) = (x-a-b)^2.$$

$$15. (a-b)(x-c) - (b-c)(x-a) - (c-a)(x-b) = 0$$

$$16. \frac{x^2-x+1}{x-1} + \frac{x^2+x+1}{x+1} = 2x.$$

$$17. \frac{4}{x+2} + \frac{7}{x+3} = \frac{37}{x^2+5x+6}.$$

$$18. (x+1)^2 = x[6-(1-x)] - 2.$$

$$19. \frac{25-\frac{1}{2}x}{x+1} + \frac{16x+4\frac{1}{2}}{3x+2} = \frac{23}{x+1} + 5.$$

$$20. \frac{3abc}{a+b} + \frac{a^2b^2}{(a+b)^2} + \frac{(2a+b)b^2x}{a(a+b)^2} = 3cx + \frac{bx}{a}.$$

$$21. \frac{4}{x-8} + \frac{3}{2x-16} - \frac{29}{24} = \frac{2}{3x-24}.$$

$$22. 5-x\left(\frac{7}{2}-\frac{2}{x}\right) = \frac{x}{2} - \frac{3x-(4-5x)}{4}.$$

$$23. \frac{1}{5} - \frac{3}{x-1} = \frac{2+\frac{x+4}{1-x}}{3}.$$

$$24. \frac{x-\frac{3}{2}}{\frac{2}{3}(x-1)} + \frac{x-\frac{5}{2}}{\frac{2}{3}(x+1)} = 1 + \frac{1}{15\left(1-\frac{1}{x^2}\right)}.$$

CHAPTER X.

PROBLEMS.

EXERCISE 67.

Ex. Find the number the sum of whose third and fourth parts is equal to 12.

Let x = the number.

Then $\frac{x}{3}$ = the third part of the number,

and $\frac{x}{4}$ = the fourth part of the number.

$\therefore \frac{x}{3} + \frac{x}{4}$ = the sum of the two parts.

But 12 = the sum of the two parts.

$\therefore \frac{x}{3} + \frac{x}{4} = 12.$

Multiply both sides by 12 :

$$4x + 3x = 144,$$

$$7x = 144.$$

$$\therefore x = 20\frac{4}{7}.$$

1. Find the number whose third and fourth parts together make 14.
2. Find the number whose third part exceeds its fourth part by 14.
3. The half, fourth, and fifth of a certain number are together equal to 76; find the number.
4. Find the number whose double exceeds its half by 12.
5. Divide 60 into two such parts that a seventh of one part may be equal to an eighth of the other.

6. Divide 50 into two such parts that a fourth of one part increased by five-sixths of the other part may be equal to 40.
7. Divide 100 into two such parts that a fourth of one part diminished by a third of the other part may be equal to 11.
8. The sum of the fourth, fifth, and sixth parts of a certain number exceeds the half of the number by 112. What is the number?
9. The sum of two numbers is 5760, and their difference is equal to one-third of the greater. What are the numbers?
10. Divide 45 into two such parts that the first part divided by 2 shall be equal to the second part multiplied by 2.
11. Find a number such that the sum of its fifth and its seventh parts shall exceed the difference of its fourth and its seventh parts by 99.
12. In a mixture of wine and water, the wine was 25 gallons more than half of the mixture, and the water 5 gallons less than one-third of the mixture. How many gallons were there of each?
13. In a certain weight of gunpowder the saltpetre was 6 pounds more than half of the weight, the sulphur 5 pounds less than the third, and the charcoal 3 pounds less than the fourth of the weight. How many pounds were there of each?
14. Divide 46 into two parts such that if one part be divided by 7, and the other by 3, the sum of the quotients shall be 10.

15. A house and garden cost \$850, and five times the price of the house was equal to twelve times the price of the garden. What was the price of each?
16. A man left the half of his property to his wife, a sixth to each of his two children, a twelfth to his brother, and the remainder, amounting to \$600, to his sister. What was the amount of his property?
17. The sum of two numbers is a and their difference is b ; find the numbers.
18. Find two numbers of which the sum is 70, such that the first divided by the second gives 2 as a quotient and 1 as a remainder.
19. Find two numbers of which the difference is 25, such that the second divided by the first gives 4 as a quotient and 4 as a remainder.
20. Divide the number 208 into two parts such that the sum of the fourth of the greater and the third of the smaller is less by 4 than four times the difference of the two parts.
21. Find four consecutive numbers whose sum is 82.

NOTE I. It is to be remembered that if x represents a person's age at the present time, his age a years ago will be represented by $x - a$, and a years hence by $x + a$.

Ex. In eight years a boy will be three times as old as he was eight years ago. How old is he?

Let x = the number of years of his age.

Then $x - 8$ = the number of years of his age eight years ago,
and $x + 8$ = the number of years of his age eight years hence

$$\begin{aligned}\therefore x + 8 &= 3(x - 8), \\ x + 8 &= 3x - 24, \\ x - 3x &= -24 - 8, \\ -2x &= -32, \\ x &= 16.\end{aligned}$$

22. A is 72 years old, and B's age is two-thirds of A's. How long is it since A was five times as old as B?
23. A mother is 70 years old, her daughter is half that age. How long is it since the mother was three and one-third times as old as the daughter?
24. A father is three times as old as the son; four years ago the father was four times as old as the son then was. What is the age of each?
25. A is twice as old as B, and seven years ago their united ages amounted to as many years as now represent the age of A. Find the ages of A and B.
26. The sum of the ages of a father and son is half what it will be in 25 years; the difference is one-third what the sum will be in 20 years. What is the age of each?

NOTE II. If A can do a piece of work in x days, the part of the work that he can do in one day will be represented by $\frac{1}{x}$. Thus, if he can do the work in 5 days, in 1 day he can do $\frac{1}{5}$ of the work.

Ex. A can do a piece of work in 5 days, and B can do it in 4 days. How long will it take A and B together to do the work?

Let x = the number of days it will take A and B together.
 Then $\frac{1}{x}$ = the part they can do in one day.
 Now, $\frac{1}{5}$ = the part A can do in one day,
 and $\frac{1}{4}$ = the part B can do in one day.
 $\therefore \frac{1}{5} + \frac{1}{4}$ = the part A and B can do in one day together.

$$\therefore \frac{1}{5} + \frac{1}{4} = \frac{1}{x},$$

$$4x + 5x = 20,$$

$$9x = 20,$$

$$x = 2\frac{2}{9}.$$

Therefore they will do the work in $2\frac{2}{9}$ days.

27. A can do a piece of work in 5 days, B in 6 days, and C in $7\frac{1}{2}$ days; in what time will they do it, all working together?

28. A can do a piece of work in $2\frac{1}{2}$ days, B in $3\frac{1}{2}$ days, and C in $3\frac{3}{4}$ days; in what time will they do it, all working together?
29. Two men who can separately do a piece of work in 15 days and 16 days, can, with the help of another, do it in 6 days. How long would it take the third man to do it alone?
30. A can do half as much work as B, B can do half as much as C, and together they can complete a piece of work in 24 days. In what time can each alone complete the work?
31. A does $\frac{2}{3}$ of a piece of work in 10 days, when B comes to help him, and they finish the work in 3 days more.. How long would it have taken B alone to do the whole work?
32. A and B together can reap a field in 12 hours, A and C in 16 hours, and A by himself in 20 hours. In what time can B and C together reap it? In what time can A, B, and C together reap it?
33. A and B together can do a piece of work in 12 days, A and C in 15 days, B and C in 20 days. In what time can they do it, all working together?
- NOTE III. If a pipe can fill a vessel in x hours, the part of the vessel filled by it in one hour will be represented by $\frac{1}{x}$. Thus, if a pipe will fill a vessel in 3 hours, in 1 hour it will fill $\frac{1}{3}$ of the vessel.
34. A tank can be filled by two pipes in 24 minutes and 30 minutes, respectively, and emptied by a third in 20 minutes. In what time will it be filled if all three are running together?
35. A tank can be filled in 15 minutes by two pipes, A and B, running together. After A has been running by

itself for 5 minutes, B is also turned on, and the tank is filled in 13 minutes more. In what time may it be filled by each pipe separately?

36. A cistern could be filled by two pipes in 6 hours and 8 hours, respectively, and could be emptied by a third in 12 hours. In what time would the cistern be filled if the pipes were all running together?
37. A tank can be filled by three pipes in 1 hour and 20 minutes, 3 hours and 20 minutes, and 5 hours, respectively. In what time will the tank be filled when all three pipes are running together?
38. If three pipes can fill a cistern in a , b , and c minutes, respectively, in what time will it be filled by all three running together?
39. The capacity of a cistern is $755\frac{1}{2}$ gallons. The cistern has three pipes, of which the first lets in 12 gallons in $3\frac{1}{2}$ minutes, the second $15\frac{1}{2}$ gallons in $2\frac{1}{2}$ minutes, the third 17 gallons in 3 minutes. In what time will the cistern be filled by the three pipes running together?

NOTE IV. In questions involving distance, time, and rate:

$$\frac{\text{Distance}}{\text{Rate}} = \text{Time}.$$

Thus, if a man travels 40 miles at the rate of 4 miles an hour,

$$\frac{40}{4} = \text{the number of hours required}.$$

- Ex. A courier who goes at the rate of $31\frac{1}{2}$ miles in 5 hours, is followed, after 8 hours, by another who goes at the rate of $22\frac{1}{2}$ miles in 3 hours. In how many hours will the second overtake the first?

Since the first goes $31\frac{1}{2}$ miles in 5 hours, his rate per hour is $6\frac{1}{2}$ miles.

Since the second goes $22\frac{1}{2}$ miles in 3 hours, his rate per hour is $7\frac{1}{2}$ miles.

Let x = the number of hours the first is travelling.

Then $x - 8$ = the number of hours the second is travelling.

Then $6\frac{2}{3}x$ = the number of miles the first travels ;

$(x - 8) 7\frac{1}{2}$ = the number of miles the second travels.

They both travel the same distance.

$$\therefore 6\frac{2}{3}x = (x - 8) 7\frac{1}{2}.$$

The solution of which gives 42 hours.

40. A sets out and travels at the rate of 7 miles in 5 hours. Eight hours afterwards, B sets out from the same place and travels in the same direction, at the rate of 5 miles in 3 hours. In how many hours will B overtake A?
41. A person walks to the top of a mountain at the rate of $2\frac{1}{2}$ miles an hour, and down the same way at the rate of $3\frac{1}{2}$ miles an hour, and is out 5 hours. How far is it to the top of the mountain?
42. A person has a hours at his disposal. How far may he ride in a coach which travels b miles an hour, so as to return home in time, walking back at the rate of c miles an hour?
43. The distance between London and Edinburgh is 360 miles. One traveller starts from Edinburgh and travels at the rate of 10 miles an hour; another starts at the same time from London, and travels at the rate of 8 miles an hour. How far from London will they meet?
44. Two persons set out from the same place in opposite directions. The rate of one of them per hour is a mile less than double that of the other, and in 4 hours they are 32 miles apart. Determine their rates.

45. In going a certain distance, a train travelling 35 miles an hour takes 2 hours less than one travelling 25 miles an hour. Determine the distance.

NOTE V. In problems relating to clocks, it is to be observed that the minute-hand moves *twelve times* as fast as the hour-hand.

Ex. Find the time between two and three o'clock when the hands of a clock are :

- I. Together.
- II. At right angles to each other.
- III. Opposite to each other.

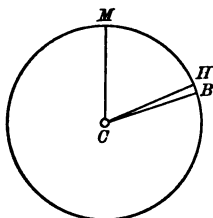


Fig. 1.

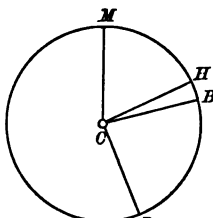


Fig. 2.

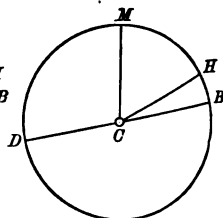


Fig. 3.

I. Let CH and CM (Fig. 1) denote the positions of the hour and minute hands at 2 o'clock, and CB the position of both hands when together.

Then arc HB = *one-twelfth* of arc MB .

Let x = the number of minute-spaces in arc MB .

Then $\frac{x}{12}$ = the number of minute-spaces in arc HB ,

and 10 = the number of minute-spaces in arc MH .

Now arc MB = the arc MH + arc HB .

That is, $x = 10 + \frac{x}{12}$.

The solution of this equation gives $x = 10\frac{1}{11}$.

Hence, the time is $10\frac{1}{11}$ minutes past 2 o'clock.

II. Let CB and CD (Fig. 2) denote the positions of the hour and minute hands when at right angles to each other.

Let x = the number of minute-spaces in arc $MHBD$.

Then $\frac{x}{12}$ = the number of minute-spaces in arc HB ,

and 10 = the number of minute-spaces in arc MH .

15 = the number of minute-spaces in arc BD .

Now arc $MHBD$ = the arcs $MH + HB + BD$.

That is, $x = 10 + \frac{x}{12} + 15$.

The solution of this equation gives $x = 27\frac{3}{4}$.

Hence, the time is $27\frac{3}{4}$ minutes past 2 o'clock.

III. Let CB and CD (Fig. 3) denote the positions of the hour and minute hands when opposite to each other.

Let x = the number of minute-spaces in arc $MHBD$.

Then $\frac{x}{12}$ = the number of minute-spaces in arc HB ,

and 10 = the number of minute-spaces in arc MH ,

30 = the number of minute-spaces in arc BD .

Now arc $MHBD$ = the arcs $MH + HB + BD$.

That is, $x = 10 + \frac{x}{12} + 30$.

The solution of this equation gives $x = 43\frac{7}{11}$.

Hence, the time is $43\frac{7}{11}$ minutes past 2 o'clock.

46. At what time are the hands of a watch together :

- I. Between 3 and 4 ?
- II. Between 6 and 7 ?
- III. Between 9 and 10 ?

47. At what time are the hands of a watch at right-angles :

- I. Between 3 and 4 ?
- II. Between 4 and 5 ?
- III. Between 7 and 8 ?

48. At what time are the hands of a watch opposite to each other :

- I. Between 1 and 2 ?
- II. Between 4 and 5 ?
- III. Between 8 and 9 ?

49. It is between 2 and 3 o'clock; but a person looking at his watch and mistaking the hour-hand for the minute-hand, fancies that the time of day is 55 minutes earlier than it really is. What is the true time?

NOTE VI. It is to be observed that if a represent the number of feet in the length of a step or leap, and x the number of steps or leaps taken, then ax will represent the number of feet in the distance made.

- Ex. A hare takes 4 leaps to a greyhound's 3; but 2 of the greyhound's leaps are equivalent to 3 of the hare's. The hare has a start of 50 leaps. How many leaps must the greyhound take to catch the hare?

Let $3x =$ the number of leaps taken by the greyhound.

Then $4x =$ the number of leaps of the hare in the same time.

Also, let a denote the number of feet in one leap of the hare.

Then $\frac{3a}{2}$ will denote the number of feet in one leap of the greyhound.

That is, $3x \times \frac{3a}{2} =$ the number of feet in whole distance,

and $(50 + 4x)a =$ the number of feet in whole distance.

$$\therefore \frac{9ax}{2} = (50 + 4x)a.$$

$$\text{Divide by } a, \quad \frac{9x}{2} = 50 + 4x,$$

$$9x = 100 + 8x.$$

$$x = 100.$$

$$\therefore 3x = 300.$$

Thus the greyhound must take 300 leaps.

50. A hare takes 6 leaps to a dog's 5, and 7 of the dog's leaps are equivalent to 9 of the hare's. The hare has a start of 50 of her own leaps. How many leaps will the hare take before she is caught?

51. A greyhound makes 3 leaps while a hare makes 4; but 2 of the greyhound's leaps are equivalent to 3 of the hare's. The hare has a start of 50 of the greyhound's leaps. How many leaps does each take before the hare is caught?
52. A greyhound makes 2 leaps while a hare makes 3; but 1 leap of the greyhound is equivalent to 2 of the hare's. The hare has a start of 80 of her own leaps. How many leaps will the hare take before she is caught?

NOTE VII. It is to be observed that if the number of units in the breadth and length of a rectangle be represented by x and $x + a$, respectively, then $x(x + a)$ will represent the number of surface units in the rectangle, the unit of surface having the same name as the linear unit in which the sides of the rectangle are expressed.

53. A rectangle whose length is 5 feet more than its breadth would have its area increased by 22 feet if its length and breadth were each made a foot more. Find its dimensions.
54. A rectangle has its length and breadth respectively 5 feet longer and 3 feet shorter than the side of the equivalent square. Find its area.
55. The length of a rectangle is an inch less than double its breadth; and when a strip 3 inches wide is cut off all round, the area is diminished by 210 inches. Find the size of the rectangle at first.
56. The length of a floor exceeds the breadth by 4 feet; if each dimension were increased by 1 foot, the area of the room would be increased by 27 square feet. Find its dimensions.

NOTE VIII. It is to be observed that if b pounds of metal lose a pounds when weighed in water, 1 pound will lose $\frac{1}{b}$ of a pounds, or $\frac{a}{b}$ of a pound.

57. A mass of tin and lead weighing 180 pounds loses 21 pounds when weighed in water; and it is known that 37 pounds of tin lose 5 pounds, and 23 pounds of lead lose 2 pounds, when weighed in water. How many pounds of tin and of lead in the mass?
58. If 19 pounds of gold lose 1 pound, and 10 pounds of silver lose 1 pound, when weighed in water, find the amount of each in a mass of gold and silver weighing 106 pounds in air and 99 pounds in water.
59. Fifteen sovereigns should weigh 77 pennyweights; but a parcel of light sovereigns, having been weighed and counted, was found to contain 9 more than was supposed from the weight; and it appeared that 21 of these coins weighed the same as 20 true sovereigns. How many were there all together?
-
60. There are two silver cups, and one cover for both. The first weighs 12 ounces, and with the cover weighs twice as much as the other without it; but the second with the cover weighs one-third more than the first without it. Find the weight of the cover.
61. A man wishes to enclose a circular piece of ground with palisades, and finds that if he sets them a foot apart he will have too few by 150; but if he sets them a yard apart he will have too many by 70. What is the circuit of the piece of ground?
62. A horse was sold at a loss for \$200; but if it had been sold for \$250, the gain would have been three-fourths of the loss when sold for \$200. Find the value of the horse.
63. A and B shoot by turns at a target. A puts 7 bullets out of 12, and B 9 out of 12, into the centre. Between them they put in 32 bullets. How many shots did each fire?

64. A boy buys a number of apples at the rate of 5 for 2 pence. He sells half of them at 2 a penny and the rest at 3 a penny, and clears a penny by the transaction. How many does he buy?
65. A person bought a piece of land for \$6750, of which he kept $\frac{1}{3}$ for himself. At the cost of \$250 he made a road which took $\frac{1}{10}$ of the remainder, and then sold the rest at $12\frac{1}{2}$ cents a square yard more than double the price it cost him, thus clearing his outlay and \$500 besides. How much land did he buy, and what was the cost price per yard?
66. A boy who runs at the rate of 12 yards per second starts 20 yards behind another whose rate is $10\frac{1}{2}$ yards per second. How soon will the first boy be 10 yards ahead of the second?
67. A merchant adds yearly to his capital one-third of it, but takes from it at the end of each year, \$5000 for expenses. At the end of the third year, after deducting the last \$5000, he has twice his original capital. How much had he at first?
68. A shepherd lost a number of sheep equal to one-fourth of his flock and one-fourth of a sheep; then, he lost a number equal to one-third of what he had left and one-third of a sheep; finally he lost a number equal to one-half of what now remained and one-half a sheep, after which he had but 25 sheep left. How many had he at first?
69. A trader maintained himself for three years at an expense of \$250 a year; and each year increased that part of his stock which was not so expended by one-third of it. At the end of the third year his original stock was doubled. What was his original stock?

70. A cask contains 12 gallons of wine and 18 gallons of water; another cask contains 9 gallons of wine and 3 gallons of water. How many gallons must be drawn from each cask to produce a mixture containing 7 gallons of wine and 7 gallons of water?
71. The members of a club subscribe each as many dollars as there are members. If there had been 12 more members, the subscription from each would have been \$10 less, to amount to the same sum. How many members were there?
72. A number of troops being formed into a solid square, it was found there were 60 men over; but when formed in a column with 5 men more in front than before, and 3 men less in depth, there was lacking one man to complete it. Find the number of troops.
73. An officer can form the men of his regiment into a hollow square twelve deep. The number of men in the regiment is 1296. Find the number of men in front of the hollow square.
74. A person starts from P and walks towards Q at the rate of 3 miles an hour; 20 minutes later another person starts from Q and walks towards P at the rate of 4 miles an hour. The distance from P to Q is 20 miles. How far from P will they meet?
75. A person engaged to work a days on these conditions: for each day he worked he was to receive b cents, and for each day he was idle he was to forfeit c cents. At the end of a days he received d cents. How many days was he idle?
76. A banker has two kinds of coins: it takes a pieces of the first to make a dollar, and b pieces of the second to make a dollar. A person wishes to obtain c pieces for a dollar. How many pieces of each kind must the banker give him?

CHAPTER XI.

SIMULTANEOUS EQUATIONS OF THE FIRST DEGREE.

184. If *one* equation contain *two* unknown numbers, an *indefinite number of pairs* of values may be found that will satisfy the equation.

Thus, in the equation $x + y = 10$, *any values* may be given to x , and *corresponding values* for y may be found. *Any pair* of these values substituted for x and y will satisfy the equation.

185. But if a second equation be given, expressing *different relations* between the unknown numbers, only *one pair* of values of x and y can be found that will satisfy *both* equations.

Thus, if besides the equation $x + y = 10$, another equation, $x - y = 2$, be given, it is evident that the values of x and y which will satisfy both equations are

$$\left. \begin{array}{l} x=6 \\ y=4 \end{array} \right\},$$

for $6 + 4 = 10$, and $6 - 4 = 2$; and these are the *only* values of x and y that will satisfy *both* equations.

186. Equations that express *different* relations between the unknown numbers are called **independent equations**.

Thus, $x + y = 10$ and $x - y = 2$ are independent equations; they express *different* relations between x and y . But $x + y = 10$ and $3x + 3y = 30$ are not independent

equations; one is derived immediately from the other, and both express the *same* relation between the unknown numbers.

187. Equations that are to be satisfied by the *same values* of the unknown numbers are called **simultaneous equations**.

188. Simultaneous equations are solved by combining the equations so as to obtain a single equation containing only one unknown number; and this process is called **elimination**.

Three methods of elimination are generally given:

- I. By Addition or Subtraction.
- II. By Substitution.
- III. By Comparison.

ELIMINATION BY ADDITION OR SUBTRACTION.

$$\begin{array}{rcl} \text{(1) Solve:} & 2x - 3y = 4 & \text{(1)} \\ & 3x + 2y = 32 & \text{(2)} \end{array}$$

Multiply (1) by 2 and (2) by 3,

$$4x - 6y = 8 \quad (3)$$

$$9x + 6y = 96 \quad (4)$$

$$\text{Add (3) and (4),} \quad \frac{13x}{} = 104$$

$$\therefore x = 8.$$

Substitute the value of x in (2),

$$24 + 2y = 32.$$

$$\therefore y = 4.$$

In this solution y is eliminated by *addition*.

$$\begin{array}{rcl} \text{(2) Solve:} & 6x + 35y = 177 & \text{(1)} \\ & 8x - 21y = 33 & \text{(2)} \end{array}$$

Multiply (1) by 4 and (2) by 3,

$$24x + 140y = 708 \quad (3)$$

$$24x - 63y = 99 \quad (4)$$

$$\text{Subtract (4) from (3),} \quad \frac{203y}{} = 609$$

$$\therefore y = 3.$$

Substitute the value of y in (2).

$$8x - 63 = 33,$$

$$\therefore x = 12.$$

In this solution x is eliminated by *subtraction*.

189. Hence, to eliminate an unknown number by addition or subtraction,

Multiply the equations by such numbers as will make the coefficients of the symbol for this unknown number equal in the resulting equations.

Add the resulting equations, or subtract one from the other, according as these equal coefficients have unlike or like signs.

NOTE. It is generally best to select that unknown number to be eliminated which requires the smallest multipliers to make its coefficients equal; and the smallest multiplier for each equation is found by dividing the L. C. M. of the coefficients of this unknown number by the given coefficient in that equation. Thus, in example (2), the L. C. M. of 6 and 8 (the coefficients of x), is 24, and hence the smallest multipliers of the two equations are 4 and 3 respectively.

Sometimes the solution is simplified by first adding the given equations, or by subtracting one from the other.

(3)	$x + 40y = 51$	(1)
	$40x + y = 99$	(2)
Add (1) and (2),	$50x + 50y = 150$	(3)
Divide (3) by 50,	$x + y = 3.$	(4)
Subtract (4) from (1),	$48y = 48.$	
	$\therefore y = 1.$	
Subtract (4) from (2),	$48x = 96.$	
	$\therefore x = 2.$	

EXERCISE 68.

Solve by addition or subtraction :

1. $2x + 3y = 7$ }	3. $7x + 2y = 30$ }	5. $5x + 4y = 58$ }
$4x - 5y = 3$ }	$y - 3x = 2$ }	$3x + 7y = 67$ }
2. $x - 2y = 4$ }	4. $3x - 5y = 51$ }	6. $3x + 2y = 39$ }
$2x - y = 5$ }	$2x + 7y = 3$ }	$3y - 2x = 13$ }

$$\begin{array}{l} 7. \quad 3x - 4y = -5 \\ \quad \quad 4x - 5y = 1 \end{array} \}$$

$$\begin{array}{l} 11. \quad 12x + 7y = 176 \\ \quad \quad 3y - 19x = 3 \end{array} \}$$

$$\begin{array}{l} 8. \quad 11x + 3y = 100 \\ \quad \quad 4x - 7y = 4 \end{array} \}$$

$$\begin{array}{l} 12. \quad 2x - 7y = 8 \\ \quad \quad 4y - 9x = 19 \end{array} \}$$

$$\begin{array}{l} 9. \quad x + 49y = 693 \\ \quad \quad 49x + y = 357 \end{array} \}$$

$$\begin{array}{l} 13. \quad 69y - 17x = 103 \\ \quad \quad 14x - 13y = -41 \end{array} \}$$

$$\begin{array}{l} 10. \quad 17x + 3y = 57 \\ \quad \quad 16y - 3x = 23 \end{array} \}$$

$$\begin{array}{l} 14. \quad 17x + 30y = 59 \\ \quad \quad 19x + 28y = 77 \end{array} \}$$

ELIMINATION BY SUBSTITUTION.

Solve :

$$\begin{array}{l} 2x + 3y = 8 \\ 3x + 7y = 7 \end{array} \}$$

$$2x + 3y = 8 \quad (1)$$

$$3x + 7y = 7 \quad (2)$$

$$\text{Transpose } 3y \text{ in (1),} \quad 2x = 8 - 3y. \quad (3)$$

$$\text{Divide by coefficient of } x, \quad x = \frac{8 - 3y}{2} \quad (4)$$

Substitute the value of x in (2), $3\left(\frac{8 - 3y}{2}\right) + 7y = 7,$

$$\frac{24 - 9y}{2} + 7y = 7,$$

$$24 - 9y + 14y = 14,$$

$$5y = -10.$$

$$\therefore y = -2.$$

Substitute the value of y in (1),

$$2x - 6 = 8.$$

$$\therefore x = 7.$$

190. Hence to eliminate an unknown number by substitution,

From one of the equations obtain the value of one of the unknown numbers in terms of the other.

Substitute for this unknown number its value in the other equation, and reduce the resulting equation.

EXERCISE 69.

Solve by substitution :

1.
$$\begin{cases} 3x - 4y = 2 \\ 7x - 9y = 7 \end{cases}$$

8.
$$\begin{cases} 3x - 4y = 18 \\ 3x + 2y = 0 \end{cases}$$

2.
$$\begin{cases} 7x - 5y = 24 \\ 4x - 3y = 11 \end{cases}$$

9.
$$\begin{cases} 9x - 5y = 52 \\ 8y - 3x = 8 \end{cases}$$

3.
$$\begin{cases} 3x + 2y = 32 \\ 20x - 3y = 1 \end{cases}$$

10.
$$\begin{cases} 5x - 3y = 4 \\ 12y - 7x = 10 \end{cases}$$

4.
$$\begin{cases} 11x - 7y = 37 \\ 8x + 9y = 41 \end{cases}$$

11.
$$\begin{cases} 9y - 7x = 13 \\ 15x - 7y = 9 \end{cases}$$

5.
$$\begin{cases} 7x + 5y = 60 \\ 13x - 11y = 10 \end{cases}$$

12.
$$\begin{cases} 5x - 2y = 51 \\ 19x - 3y = 180 \end{cases}$$

6.
$$\begin{cases} 6x - 7y = 42 \\ 7x - 6y = 75 \end{cases}$$

13.
$$\begin{cases} 4x + 9y = 106 \\ 8x + 17y = 198 \end{cases}$$

7.
$$\begin{cases} 10x + 9y = 290 \\ 12x - 11y = 130 \end{cases}$$

14.
$$\begin{cases} 8x + 3y = 3 \\ 12x + 9y = 3 \end{cases}$$

ELIMINATION BY COMPARISON.

Solve :

$$\begin{cases} 2x - 9y = 11 \\ 3x - 4y = 7 \end{cases}$$

$$2x - 9y = 11, \quad (1)$$

$$3x - 4y = 7. \quad (2)$$

Transpose $9y$ in (1) and $4y$ in (2), $2x = 11 + 9y,$ (3)

$$3x = 7 + 4y. \quad (4)$$

Divide (3) by 2 and (4) by 3,

$$x = \frac{11 + 9y}{2}, \quad (5)$$

$$x = \frac{7 + 4y}{3}. \quad (6)$$

Equate the values of x ,

$$\frac{11 + 9y}{2} = \frac{7 + 4y}{3}. \quad (7)$$

Reduce (7),

$$33 + 27y = 14 + 8y,$$

$$19y = -19.$$

$$\therefore y = -1.$$

Substitute the value of y in (1), $2x + 9 = 11.$

$$\therefore x = 1.$$

191. Hence, to eliminate an unknown number by comparison,

From each equation obtain the value of one of the unknown numbers in terms of the other.

Form an equation from these equal values and reduce the equation.

NOTE. If, in the last example, (3) be divided by (4), the resulting equation, $\frac{2}{3} = \frac{11 + 9y}{7 + 4y}$, would, when reduced, give the value of y . This is the shortest method, and therefore to be preferred.

EXERCISE 70.

Solve by comparison :

$$\begin{array}{l} 1. \quad x + 15y = 53 \\ \quad \quad 3x + \quad y = 27 \end{array} \}$$

$$\begin{array}{l} 2. \quad 4x + 9y = 51 \\ \quad \quad 8x - 13y = 9 \end{array} \}$$

$$\begin{array}{l} 3. \quad 4x + 3y = 48 \\ \quad \quad 5y - 3x = 22 \end{array} \}$$

$$\begin{array}{l} 4. \quad 2x + 3y = 43 \\ \quad \quad 10x - \quad y = 7 \end{array} \}$$

$$\begin{array}{l} 5. \quad 5x - 7y = 33 \\ \quad \quad 11x + 12y = 100 \end{array} \}$$

$$\begin{array}{l} 6. \quad 5x + 7y = 43 \\ \quad \quad 11x + 9y = 69 \end{array} \}$$

$$\begin{array}{l} 7. \quad 8x - 21y = 33 \\ \quad \quad 6x + 35y = 177 \end{array} \}$$

$$\begin{array}{l} 8. \quad 3y - 7x = 4 \\ \quad \quad 2y + 5x = 22 \end{array} \}$$

$$\begin{array}{l} 9. \quad 21y + 20x = 165 \\ \quad \quad 77y - 30x = 295 \end{array} \}$$

$$\begin{array}{l} 10. \quad 11x - 10y = 14 \\ \quad \quad 5x + 7y = 41 \end{array} \}$$

$$\begin{array}{l} 11. \quad 7y - 3x = 139 \\ \quad \quad 2x + 5y = 91 \end{array} \}$$

$$\begin{array}{l} 12. \quad 17x + 12y = 59 \\ \quad \quad 19x - 4y = 153 \end{array} \}$$

$$\begin{array}{l} 13. \quad 24x + 7y = 27 \\ \quad \quad 8x - 33y = 115 \end{array} \}$$

$$\begin{array}{l} 14. \quad x = 3y - 19 \\ \quad \quad y = 3x - 23 \end{array} \}$$

192. Each equation must be simplified, if necessary, before the elimination is performed.

$$\text{Solve: } \left. \begin{aligned} (x-1)(y+2) &= (x-3)(y-1) + 8 \\ \frac{2x-1}{5} - \frac{3(y-2)}{4} &= 1 \end{aligned} \right\}$$

$$(x-1)(y+2) = (x-3)(y-1) + 8 \quad (1)$$

$$\frac{2x-1}{5} - \frac{3(y-2)}{4} = 1 \quad (2)$$

$$\text{Simplify (1),} \quad xy + 2x - y - 2 = xy - x - 3y + 3 + 8.$$

$$\text{Transpose and combine,} \quad 3x + 2y = 13. \quad (3)$$

$$\text{Simplify (2),} \quad 8x - 4 - 15y + 30 = 20.$$

$$\text{Transpose and combine,} \quad 8x - 15y = -6. \quad (4)$$

$$\text{Multiply (3) by 8,} \quad 24x + 16y = 104. \quad (5)$$

$$\text{Multiply (4) by 3,} \quad 24x - 45y = -18. \quad (6)$$

$$\text{Subtract (6) from (5),} \quad 61y = 122.$$

$$\therefore y = 2.$$

$$\text{Substitute the value of } y \text{ in (3),} \quad 3x + 4 = 13.$$

$$\therefore x = 3.$$

EXERCISE 71.

Solve:

$$1. \left. \begin{aligned} x(y+7) &= y(x+1) \\ 2x+20 &= 3y+1 \end{aligned} \right\} \quad 3. \left. \begin{aligned} \frac{2}{x+3} &= \frac{3}{y-2} \\ 5(x+3) &= 3(y-2) + 2 \end{aligned} \right\}$$

$$2. \left. \begin{aligned} 2x - \frac{y-3}{5} - 4 &= 0 \\ 3y + \frac{x-2}{3} - 9 &= 0 \end{aligned} \right\} \quad 4. \left. \begin{aligned} \frac{x-4}{5} - \frac{y+2}{10} &= 0 \\ \frac{x}{6} + \frac{y-2}{4} &= 3 \end{aligned} \right\}$$

$$5. \left. \begin{aligned} (x+1)(y+2) - (x+2)(y+1) &= -1 \\ 3(x+3) - 4(y+4) &= -8 \end{aligned} \right\}$$

$$6. \left. \begin{aligned} \frac{x-2}{5} - \frac{10-x}{3} &= \frac{y-10}{4} \\ \frac{2y+4}{3} - \frac{2x+y}{8} &= \frac{x+13}{4} \end{aligned} \right\}$$

$$7. \left. \begin{aligned} \frac{x+1}{3} - \frac{y+2}{4} &= \frac{2(x-y)}{5} \\ \frac{x-3}{4} - \frac{y-3}{3} &= 2y-x \end{aligned} \right\}$$

$$15. \left. \begin{aligned} \frac{x-4}{5} &= \frac{y+2}{10} \\ \frac{x}{6} + \frac{y-2}{4} &= 3 \end{aligned} \right\}$$

$$8. \left. \begin{aligned} \frac{3x-2y}{5} + \frac{5x-3y}{3} &= x+1 \\ \frac{2x-3y}{3} + \frac{4x-3y}{2} &= y+1 \end{aligned} \right\}$$

$$16. \left. \begin{aligned} \frac{3x+12y}{11} &= 9 \\ \frac{1-3x}{7} &= \frac{11-3y}{5} \end{aligned} \right\}$$

$$9. \left. \begin{aligned} \frac{2x-y+3}{3} - \frac{x-2y+3}{4} &= 4 \\ \frac{3x-4y+3}{4} + \frac{4x-2y-9}{3} &= 4 \end{aligned} \right\}$$

$$10. \left. \begin{aligned} 1\frac{1}{2}x &= 1\frac{1}{2}y + 4\frac{1}{12} \\ 4\frac{1}{2}x &= \frac{1}{2}y - 21\frac{1}{12} \end{aligned} \right\}$$

$$17. \left. \begin{aligned} 5x - \frac{1}{4}(5y+2) &= 32 \\ 3y + \frac{1}{3}(x+2) &= 9 \end{aligned} \right\}$$

$$11. \left. \begin{aligned} \frac{13}{x+2y+3} &= -\frac{3}{4x-5y+6} \\ \frac{3}{6x-5y+4} &= \frac{19}{3x+2y+1} \end{aligned} \right\}$$

$$18. \left. \begin{aligned} 3x - 0.25y &= 28 \\ 0.12x + 0.7y &= 2.54 \end{aligned} \right\}$$

$$12. \left. \begin{aligned} \frac{x+y}{y-x} &= \frac{15}{8} \\ 9x - \frac{3y+44}{7} &= 100 \end{aligned} \right\}$$

$$19. \left. \begin{aligned} 7(x-1) &= 3(y+8) \\ \frac{4x+2}{9} &= \frac{5y+9}{2} \end{aligned} \right\}$$

$$13. \left. \begin{aligned} \frac{3x-5y}{2} + 3 &= \frac{2x+y}{5} \\ 8 - \frac{x-2y}{4} &= \frac{x}{2} + \frac{y}{3} \end{aligned} \right\}$$

$$20. \left. \begin{aligned} 7x + \frac{1}{3}(2y+4) &= 16 \\ 3y - \frac{1}{4}(x+2) &= 8 \end{aligned} \right\}$$

$$14. \left. \begin{aligned} \frac{4x-3y-7}{5} &= \frac{3x}{10} - \frac{2y}{15} - \frac{5}{6} \\ \frac{y-1}{3} + \frac{x}{2} - \frac{3y}{20} - 1 &= \frac{y-x}{15} + \frac{x}{6} + \frac{1}{10} \end{aligned} \right\}$$

$$21. \left. \begin{aligned} \frac{5x-6y}{13} + 3x &= 4y-2 \\ \frac{5x+6y}{6} - \frac{3x-2y}{4} &= 2y-2 \end{aligned} \right\}$$

$$22. \left. \begin{aligned} \frac{5x-3}{2} - \frac{3x-19}{2} &= 4 - \frac{3y-x}{3} \\ \frac{2x+y}{2} - \frac{9x-7}{8} &= \frac{3(y+3)}{4} - \frac{4x+5y}{16} \end{aligned} \right\}$$

$$23. \left. \begin{aligned} 3y+11 &= \frac{4x^2-y(x+3y)}{x-y+4} + 31-4x \\ (x+7)(y-2)+3 &= 2xy-(y-1)(x+1) \end{aligned} \right\}$$

$$24. \left. \begin{aligned} \frac{6x+9}{4} + \frac{3x+5y}{4x-6} &= 3\frac{1}{4} + \frac{3x+4}{2} \\ \frac{8y+7}{10} + \frac{6x-3y}{2y-8} &= 4 + \frac{4y-9}{5} \end{aligned} \right\}$$

$$25. \left. \begin{aligned} x - \frac{2y-x}{23-x} &= 20 - \frac{59-2x}{2} \\ y + \frac{y-3}{x-18} &= 30 - \frac{73-3y}{3} \end{aligned} \right\}$$

LITERAL SIMULTANEOUS EQUATIONS.

193. The method of solving literal simultaneous equations is as follows:

$$\text{Solve:} \quad \left. \begin{aligned} ax+by &= m \\ cx+dy &= n \end{aligned} \right\}$$

$$ax+by = m \quad (1)$$

$$cx+dy = n \quad (2)$$

$$\text{Multiply (1) by } c, \quad acx+bcy = cm \quad (3)$$

$$\text{Multiply (2) by } a, \quad acx+ady = an \quad (4)$$

$$\text{Subtract (4) from (3),} \quad (bc-ad)y = cm-an$$

$$\text{Divide by coefficient of } y, \quad y = \frac{cm-an}{bc-ad}$$

To find the value of x :

$$\text{Multiply (1) by } d, \quad adx + bdy = dm \quad (5)$$

$$\text{Multiply (2) by } b, \quad bcx + bdy = bn \quad (6)$$

$$\text{Subtract (6) from (5),} \quad (ad - bc)x = dm - bn$$

$$\text{Divide by coefficient of } x, \quad x = \frac{dm - bn}{ad - bc}$$

EXERCISE 72.

Solve:

$$\begin{array}{lll} 1. \quad \left. \begin{array}{l} x + y = a \\ x - y = b \end{array} \right\} & 3. \quad \left. \begin{array}{l} mx + ny = a \\ px + qy = b \end{array} \right\} & 5. \quad \left. \begin{array}{l} mx - ny = r \\ m'x + n'y = r' \end{array} \right\} \end{array}$$

$$\begin{array}{lll} 2. \quad \left. \begin{array}{l} ax + by = c \\ px + qy = r \end{array} \right\} & 4. \quad \left. \begin{array}{l} ax + by = e \\ ax + cy = d \end{array} \right\} & 6. \quad \left. \begin{array}{l} ax + by = c \\ dx + fy = c' \end{array} \right\} \end{array}$$

$$\begin{array}{ll} 7. \quad \left. \begin{array}{l} \frac{x}{a} + \frac{y}{b} = c \\ \frac{x}{b} - \frac{y}{a} = -c \end{array} \right\} & 12. \quad \left. \begin{array}{l} \frac{x - y + 1}{x - y - 1} = a \\ \frac{x + y + 1}{x + y - 1} = b \end{array} \right\} \end{array}$$

$$\begin{array}{ll} 8. \quad \left. \begin{array}{l} abx + cdy = 2 \\ ax - cy = \frac{d - b}{bd} \end{array} \right\} & 13. \quad \left. \begin{array}{l} ax = by + \frac{a^2 + b^2}{2} \\ (a - b)x = (a + b)y \end{array} \right\} \end{array}$$

$$\begin{array}{ll} 9. \quad \left. \begin{array}{l} \frac{a}{b + y} = \frac{b}{3a + x} \\ ax + 2by = d \end{array} \right\} & 14. \quad \left. \begin{array}{l} ax + by = c^2 \\ \frac{a}{b + y} - \frac{b}{a + x} = 0 \end{array} \right\} \end{array}$$

$$\begin{array}{ll} 10. \quad \left. \begin{array}{l} \frac{x}{a + b} - \frac{y}{a - b} = \frac{1}{a + b} \\ \frac{x}{a + b} + \frac{y}{a - b} = \frac{1}{a - b} \end{array} \right\} & 15. \quad \left. \begin{array}{l} \frac{x}{a + b} + \frac{y}{a - b} = 2a \\ \frac{x - y}{2ab} = \frac{x + y}{a^2 + b^2} \end{array} \right\} \end{array}$$

$$\begin{array}{ll} 11. \quad \left. \begin{array}{l} a(a - x) = b(x + y - a) \\ a(y - b - x) = b(y - b) \end{array} \right\} & 16. \quad \left. \begin{array}{l} bx - bc = ay - ac \\ x - y = a - b \end{array} \right\} \end{array}$$

17. $\left. \begin{aligned} \frac{x-a}{y-b} &= c \\ a(x-a) + b(y-b) + abc &= 0 \end{aligned} \right\}$
18. $\left. \begin{aligned} (a+b)x - (a-b)y &= 4ab \\ (a-b)x + (a+b)y &= 2a^2 - 2b^2 \end{aligned} \right\}$
19. $\left. \begin{aligned} (x+a)(y+b) - (x-a)(y-b) &= 2(a-b)^2 \\ x-y + 2(a-b) &= 0 \end{aligned} \right\}$
20. $\left. \begin{aligned} (a+b)(x+y) - (a-b)(x-y) &= a^2 \\ (a-b)(x+y) + (a+b)(x-y) &= b^2 \end{aligned} \right\}$

194. Fractional simultaneous equations, of which the denominators are simple expressions and contain the unknown numbers, may be solved as follows:

(1) Solve:

$$\left. \begin{aligned} \frac{a}{x} + \frac{b}{y} &= m \\ \frac{c}{x} + \frac{d}{y} &= n \end{aligned} \right\}$$

$$\frac{a}{x} + \frac{b}{y} = m. \quad (1)$$

$$\frac{c}{x} + \frac{d}{y} = n. \quad (2)$$

Multiply (1) by c , $\frac{ac}{x} + \frac{bc}{y} = cm. \quad (3)$

Multiply (2) by a , $\frac{ac}{x} + \frac{ad}{y} = an. \quad (4)$

Subtract (4) from (3), $\frac{bc-ad}{y} = cm-an.$

Multiply both sides by y , $bc-ad = (cm-an)y.$
 $\therefore y = \frac{bc-ad}{cm-an}.$

Multiply (1) by d , $\frac{ad}{x} + \frac{bd}{y} = dm. \quad (5)$

Multiply (2) by b , $\frac{bc}{x} + \frac{bd}{y} = bn. \quad (6)$

Subtract (6) from (5), $\frac{ad-bc}{x} = dm - bn.$

Multiply both sides by x , $ad - bc = (dm - bn)x.$

$$\therefore x = \frac{ad-bc}{dm-bn}.$$

(2) Solve:
$$\left. \begin{aligned} \frac{5}{3x} + \frac{2}{5y} &= 7 \\ \frac{7}{6x} - \frac{1}{10y} &= 3 \end{aligned} \right\}$$

$$\frac{5}{3x} + \frac{2}{5y} = 7. \quad (1)$$

$$\frac{7}{6x} - \frac{1}{10y} = 3. \quad (2)$$

Multiply (2) by 4, $\frac{14}{3x} - \frac{2}{5y} = 12. \quad (3)$

Add (1) and (3), $\frac{19}{3x} = 19.$

Divide both sides by 19, $\frac{1}{3x} = 1.$
 $\therefore x = \frac{1}{3}.$

Substitute the value of x in (1), $5 + \frac{2}{5y} = 7.$

Transpose, $\frac{2}{5y} = 2.$

Divide both sides by 2, $\frac{1}{5y} = 1.$
 $\therefore y = \frac{1}{5}.$

Solve:

EXERCISE 73.

1. $\left. \begin{aligned} \frac{1}{x} + \frac{2}{y} &= 10 \\ \frac{4}{x} + \frac{3}{y} &= 20 \end{aligned} \right\}$

2. $\left. \begin{aligned} \frac{1}{x} + \frac{2}{y} &= a \\ \frac{3}{x} + \frac{4}{y} &= b \end{aligned} \right\}$

3. $\left. \begin{aligned} \frac{2}{x} - \frac{5}{3y} &= \frac{4}{27} \\ \frac{1}{4x} + \frac{1}{y} &= \frac{11}{72} \end{aligned} \right\}$

4. $\left. \begin{aligned} \frac{1}{x} + \frac{2}{y} &= 4 \\ \frac{3}{x} - \frac{2}{y} &= 4 \end{aligned} \right\}$

5. $\left. \begin{aligned} \frac{3}{x} - \frac{4}{y} &= 5 \\ \frac{4}{x} - \frac{5}{y} &= 6 \end{aligned} \right\}$

6. $\left. \begin{aligned} \frac{a}{x} + \frac{b}{y} &= \frac{ac}{b} \\ \frac{b}{x} + \frac{a}{y} &= \frac{bc}{a} \end{aligned} \right\}$

$$\begin{array}{lcl}
 7. \left. \begin{array}{l} \frac{2}{ax} + \frac{3}{by} = 5 \\ \frac{5}{ax} - \frac{2}{by} = 3 \end{array} \right\} & 8. \left. \begin{array}{l} \frac{m}{nx} + \frac{n}{my} = m + n \\ \frac{n}{x} + \frac{m}{y} = m^2 + n^2 \end{array} \right\} & 9. \left. \begin{array}{l} \frac{a}{x} + \frac{b}{y} = m \\ \frac{b}{x} - \frac{a}{y} = n \end{array} \right\}
 \end{array}$$

195. If three simultaneous equations are given, involving three unknown numbers, one of the unknown numbers must be eliminated between *two pairs* of the equations; then a second between the resulting equations.

196. Likewise, if four or more equations are given, involving four or more unknown numbers, one of the unknown numbers must be eliminated between three or more pairs of the equations; then a second between the pairs that can be formed of the resulting equations; and so on.

$$\begin{array}{rcl}
 \text{Solve:} & \left. \begin{array}{l} 2x - 3y + 4z = 4 \\ 3x + 5y - 7z = 12 \\ 5x - y - 8z = 5 \end{array} \right\} & \begin{array}{l} (1) \\ (2) \\ (3) \end{array}
 \end{array}$$

Eliminate z between two pairs of these equations.

$$\text{Multiply (1) by 2,} \quad 4x - 6y + 8z = 8 \quad (4)$$

$$(3) \text{ is} \quad 5x - y - 8z = 5$$

$$\text{Add,} \quad 9x - 7y = 13 \quad (5)$$

$$\text{Multiply (1) by 7,} \quad 14x - 21y + 28z = 28$$

$$\text{Multiply (2) by 4,} \quad 12x + 20y - 28z = 48$$

$$\text{Add,} \quad 26x - y = 76 \quad (6)$$

$$\text{Multiply (6) by 7,} \quad 182x - 7y = 532 \quad (7)$$

$$(5) \text{ is} \quad 9x - 7y = 13$$

$$\text{Subtract (5) from (7),} \quad 173x = 519$$

$$\therefore x = 3.$$

$$\text{Substitute the value of } x \text{ in (6),} \quad 78 - y = 76.$$

$$\therefore y = 2.$$

$$\text{Substitute the values of } x \text{ and } y \text{ in (1),} \quad 6 - 6 + 4z = 4.$$

$$\therefore z = 1.$$

EXERCISE 74.

Solve:

$$\left. \begin{array}{l} 1. \quad 5x + 3y - 6z = 4 \\ \quad 3x - y + 2z = 8 \\ \quad x - 2y + 2z = 2 \end{array} \right\}$$

$$\left. \begin{array}{l} 10. \quad 3x - y + z = 17 \\ \quad 5x + 3y - 2z = 10 \\ \quad 7x + 4y - 5z = 3 \end{array} \right\}$$

$$\left. \begin{array}{l} 2. \quad 4x - 5y + 2z = 6 \\ \quad 2x + 3y - z = 20 \\ \quad 7x - 4y + 3z = 35 \end{array} \right\}$$

$$\left. \begin{array}{l} 11. \quad x + y + z = 5 \\ \quad 3x - 5y + 7z = 75 \\ \quad 9x - 11z + 10 = 0 \end{array} \right\}$$

$$\left. \begin{array}{l} 3. \quad x + y + z = 6 \\ \quad 5x + 4y + 3z = 22 \\ \quad 15x + 10y + 6z = 53 \end{array} \right\}$$

$$\left. \begin{array}{l} 12. \quad x + 2y + 3z = 6 \\ \quad 2x + 4y + 2z = 8 \\ \quad 3x + 2y + 8z = 101 \end{array} \right\}$$

$$\left. \begin{array}{l} 4. \quad 4x - 3y + z = 9 \\ \quad 9x + y - 5z = 16 \\ \quad x - 4y + 3z = 2 \end{array} \right\}$$

$$\left. \begin{array}{l} 13. \quad x - 3y - 2z = 1 \\ \quad 2x - 3y + 5z = -19 \\ \quad 5x + 2y - z = 12 \end{array} \right\}$$

$$\left. \begin{array}{l} 5. \quad 8x + 4y - 3z = 6 \\ \quad x + 3y - z = 7 \\ \quad 4x - 5y + 4z = 8 \end{array} \right\}$$

$$\left. \begin{array}{l} 14. \quad 3x - 2y = 5 \\ \quad 4x - 3y + 2z = 11 \\ \quad x - 2y - 5z = -7 \end{array} \right\}$$

$$\left. \begin{array}{l} 6. \quad 12x + 5y - 4z = 29 \\ \quad 13x - 2y + 5z = 58 \\ \quad 17x - y - z = 15 \end{array} \right\}$$

$$\left. \begin{array}{l} 15. \quad x + y = 1 \\ \quad y + z = 9 \\ \quad x + z = 5 \end{array} \right\}$$

$$\left. \begin{array}{l} 7. \quad y - x + z = -5 \\ \quad z - y - x = -25 \\ \quad x + y + z = 35 \end{array} \right\}$$

$$\left. \begin{array}{l} 16. \quad 2x - 3y = 3 \\ \quad 3y - 4z = 7 \\ \quad 4z - 5x = 2 \end{array} \right\}$$

$$\left. \begin{array}{l} 8. \quad x + y + z = 30 \\ \quad 8x + 4y + 2z = 50 \\ \quad 27x + 9y + 3z = 64 \end{array} \right\}$$

$$\left. \begin{array}{l} 17. \quad 3x - 4y + 6z = 1 \\ \quad 2x + 2y - z = 1 \\ \quad 7x - 6y + 7z = 2 \end{array} \right\}$$

$$\left. \begin{array}{l} 9. \quad 15y = 24z - 10x + 41 \\ \quad 15x = 12y - 16z + 10 \\ \quad 18x - (7z - 13) = 14y \end{array} \right\}$$

$$\left. \begin{array}{l} 18. \quad 7x - 3y = 30 \\ \quad 9y - 5z = 34 \\ \quad x + y + z = 33 \end{array} \right\}$$

$$19. \left. \begin{aligned} x + \frac{y}{2} + \frac{z}{3} &= 6 \\ y + \frac{z}{2} + \frac{x}{3} &= -1 \\ z + \frac{x}{2} + \frac{y}{3} &= 17 \end{aligned} \right\}$$

$$23. \left. \begin{aligned} \frac{3}{x} - \frac{4}{5y} + \frac{1}{z} &= 7\frac{1}{2} \\ \frac{1}{3x} + \frac{1}{2y} + \frac{2}{z} &= 10\frac{1}{4} \\ \frac{4}{5x} - \frac{1}{2y} + \frac{4}{z} &= 16\frac{1}{10} \end{aligned} \right\}$$

$$20. \left. \begin{aligned} \frac{1}{x} + \frac{2}{y} &= 5 \\ \frac{3}{y} - \frac{4}{z} &= -6 \\ \frac{3}{z} - \frac{4}{x} &= 5 \end{aligned} \right\}$$

$$24. \left. \begin{aligned} \frac{2}{x} - \frac{3}{y} + \frac{4}{z} &= 2.9 \\ \frac{5}{x} - \frac{6}{y} - \frac{7}{z} &= -10.4 \\ \frac{9}{y} + \frac{10}{z} - \frac{8}{x} &= 14.9 \end{aligned} \right\}$$

$$21. \left. \begin{aligned} \frac{1}{x} + \frac{1}{y} - \frac{1}{z} &= a \\ \frac{1}{x} - \frac{1}{y} + \frac{1}{z} &= b \\ \frac{1}{y} + \frac{1}{z} - \frac{1}{x} &= c \end{aligned} \right\} *$$

$$25. \left. \begin{aligned} \frac{2}{x} + \frac{1}{y} - \frac{3}{z} &= 0 \\ \frac{3}{z} - \frac{2}{y} - 2 &= 0 \\ \frac{1}{x} + \frac{1}{z} - \frac{4}{3} &= 0 \end{aligned} \right\}$$

$$22. \left. \begin{aligned} bx + cy &= a \\ az + cx &= b \\ ay + bx &= c \end{aligned} \right\} \dagger$$

$$26. \left. \begin{aligned} ax + by + cz &= a \\ ax - by - cz &= b \\ ax + cy + bz &= c \end{aligned} \right\}$$

$$27. \frac{2x-y}{3} = \frac{3y+2z}{4} = \frac{x-y-z}{5} = 4$$

$$28. \frac{x-y}{a} = \frac{y-z}{b} = \frac{x+z}{c} = \frac{x-a-b}{a+b+c}$$

* In example 21, subtract from the sum of the three equations each equation separately.

† In example 22, multiply the equations by a , b , and c , respectively, and from the sum of the results subtract the double of each equation separately.

CHAPTER XII.

PROBLEMS PRODUCING SIMULTANEOUS EQUATIONS.

197. It is often necessary in the solution of problems to employ two or more letters to represent the numbers to be found. In all cases the conditions must be sufficient to give just as many equations as there are unknown numbers employed.

If there are *more* equations than unknown numbers, some of them are superfluous or contradictory; if there are *less* equations than unknown numbers, the problem is indeterminate or impossible.

- (1) When the greater of two numbers is divided by the less the quotient is 4 and the remainder 3; and when the sum of the two numbers is increased by 38, and the result divided by the greater of the two numbers, the quotient is 2 and the remainder 2. Find the numbers.

$$\begin{array}{ll} \text{Let} & x = \text{the greater number,} \\ \text{and} & y = \text{the smaller number.} \\ \text{Then} & \frac{x-3}{y} = 4, \\ \text{and} & \frac{x+y+38-2}{x} = 2. \end{array}$$

From the solution of these equations, $x = 47$, and $y = 11$.

- (2) If A give B \$10, B will have three times as much money as A. If B give A \$10, A will have twice as much money as B. How much has each?

Let x = the number of dollars A has,
and y = the number of dollars B has.

Then $y + 10$ = the number of dollars B has, and $x - 10$ = the number of dollars A has after A gives \$10 to B.

$\therefore y + 10 = 3(x - 10)$, and $x + 10 = 2(y - 10)$.

From the solution of these equations, $x = 22$, and $y = 26$.

Therefore, A has \$22 and B \$26.

EXERCISE 75.

1. The sum of two numbers divided by 2 gives as a quotient 24, and the difference between them divided by 2 gives as a quotient 17. What are the numbers?
2. The number 144 is divided into three numbers. When the first is divided by the second, the quotient is 3 and the remainder 2; and when the third is divided by the sum of the other two numbers, the quotient is 2 and the remainder 6. Find the numbers.
3. Three times the greater of two numbers exceeds twice the less by 10; and twice the greater together with three times the less is 24. Find the numbers.
4. If the smaller of two numbers is divided by the greater, the quotient is 0.21 and the remainder 0.0057; but if the greater is divided by the smaller, the quotient is 4 and the remainder 0.742. What are the numbers?
5. Seven years ago the age of a father was four times that of his son; seven years hence the age of the father will be double that of the son. What are their ages?
6. The sum of the ages of a father and son is half what it will be in 25 years; the difference between their ages is one-third of what the sum will be in 20 years. What are their ages?

7. If B give A \$25, they will have equal sums of money ; but if A give B \$22, B's money will be double that of A. How much has each ?
8. A farmer sold to one person 30 bushels of wheat and 40 bushels of barley for \$67.50 ; to another person he sold 50 bushels of wheat and 30 bushels of barley for \$85. What was the price of the wheat and of the barley per bushel ?
9. If A give B \$5, he will then have \$6 less than B ; but if he receive \$5 from B, three times his money will be \$20 more than four times B's. How much has each ?
10. The cost of 12 horses and 14 cows is \$1900 ; the cost of 5 horses and 3 cows is \$650. What is the cost of a horse and a cow respectively ?

NOTE I. A fraction of which the terms are unknown may be represented by $\frac{x}{y}$.

Ex. A certain fraction becomes equal to $\frac{1}{2}$ if 3 is added to its numerator, and equal to $\frac{2}{7}$ if 3 is added to its denominator. Determine the fraction.

Let $\frac{x}{y}$ = the required fraction.

By the conditions $\frac{x+3}{y} = \frac{1}{2}$,

and $\frac{x}{y+3} = \frac{2}{7}$.

From the solution of these equations it is found that

$$x = 6,$$

$$y = 18.$$

Therefore the fraction = $\frac{6}{18}$.

11. A certain fraction becomes equal to 2 when 7 is added to its numerator, and equal to 1 when 1 is subtracted from its denominator. Determine the fraction.

12. A certain fraction becomes equal to $\frac{1}{3}$ when 7 is added to its denominator, and equal to 2 when 13 is added to its numerator. Determine the fraction.
13. A certain fraction becomes equal to $\frac{7}{8}$ when the denominator is increased by 4, and equal to $\frac{2}{3}$ when the numerator is diminished by 15. Determine the fraction.
14. A certain fraction becomes equal to $\frac{3}{4}$ if 7 is added to the numerator, and equal to $\frac{3}{8}$ if 7 is subtracted from the denominator. Determine the fraction.
15. Find two fractions with numerators 2 and 5 respectively, such that their sum is $1\frac{1}{2}$; and if their denominators are interchanged their sum is 2.
16. A fraction which is equal to $\frac{3}{4}$ is increased to $\frac{5}{6}$ when a certain number is added to both its numerator and denominator, and is diminished to $\frac{1}{2}$ when one more than the same number is subtracted from each. Determine the fraction.

NOTE II. A number consisting of *two* digits which are unknown may be represented by $10x + y$, in which x and y represent the digits of the number. Likewise, a number consisting of *three* digits which are unknown may be represented by $100x + 10y + z$, in which x , y , and z represent the digits of the number.

For example, consider any number expressed by three digits, as 364. The expression 364 means $300 + 60 + 4$; or, 100 times 3 + 10 times 6 + 4.

Ex. The sum of the two digits of a number is 8, and if 36 be added to the number the digits will be interchanged. What is the number?

Let x = the digit in the tens' place,
 and y = the digit in the units' place.
 Then $10x + y$ = the number.
 By the conditions, $x + y = 8$, (1)
 and $10x + y + 36 = 10y + x$. (2)

From (2),	$9x - 9y = -36.$
Divide by 9,	$x - y = -4.$
Add (1) and (3),	$2x = 4.$
	$\therefore x = 2.$
Subtract (3) from (1),	$2y = 12.$
	$\therefore y = 6.$

Hence, the number is 26.

17. The sum of the two digits of a number is 10, and if 54 be added to the number the digits will be interchanged. What is the number?
18. The sum of the two digits of a number is 6, and if the number be divided by the sum of the digits the quotient will be 4. What is the number?
19. A certain number is expressed by two digits, of which the first is the greater. If the number is divided by the sum of its digits the quotient is 7; if the digits are interchanged, and the resulting number diminished by 12 is divided by the difference between the two digits, the quotient is 9. What is the number?
20. If a certain number is divided by the sum of its two digits the quotient is 6 and the remainder 3; if the digits are interchanged, and the resulting number is divided by the sum of the digits, the quotient is 4 and the remainder 9. What is the number?
21. If a certain number is divided by the sum of its two digits diminished by 2, the quotient is 5 and the remainder 1; if the digits are interchanged, and the resulting number is divided by the sum of the digits increased by 2, the quotient is 5 and the remainder 8. Find the number.
22. The first of the two digits of a number is, when doubled, 3 more than the second, and the number itself is less by 6 than five times the sum of the digits. What is the number?

23. A number is expressed by three digits, of which the first and last are alike. By interchanging the digits in the units' and tens' places the number is increased by 54; but if the digits in the tens' and hundreds' places are interchanged, 9 must be added to four times the resulting number to make it equal to the original number. What is the number?
24. A number is expressed by three digits. The sum of the digits is 21; the sum of the first and second exceeds the third by 3; and if 198 be added to the number, the digits in the units' and hundreds' places will be interchanged. Find the number.
25. A number is expressed by three digits. The sum of the digits is 9; the number is equal to forty-two times the sum of the first and second digits; and the third digit is twice the sum of the other two. Find the number.
26. A certain number, expressed by three digits, is equal to forty-eight times the sum of its digits. If 198 be subtracted from the number, the digits in the units' and hundreds' places will be interchanged; and the sum of the extreme digits is equal to twice the middle digit. Find the number.

NOTE III. If a boat moves at the rate of x miles an hour in still water, and if it is on a stream that runs at the rate of y miles an hour, then

$x + y$ represents its rate *down* the stream,

$x - y$ represents its rate *up* the stream.

27. A waterman rows 30 miles and back in 12 hours. He finds that he can row 5 miles with the stream in the same time as 3 against it. Find the time he was rowing up and down respectively.

28. A crew which can pull at the rate of 12 miles an hour down the stream, finds that it takes twice as long to come up the river as to go down. At what rate does the stream flow ?
29. A man sculls down a stream, which runs at the rate of 4 miles an hour, for a certain distance in 1 hour and 40 minutes. In returning it takes him 4 hours and 15 minutes to arrive at a point 3 miles short of his starting-place. Find the distance he pulled down the stream and the rate of his pulling.
30. A person rows down a stream a distance of 20 miles and back again in 10 hours. He finds he can row 2 miles against the stream in the same time he can row 3 miles with it. Find the time of his rowing down and of his rowing up the stream ; and also the rate of the stream.

NOTE IV. When commodities are mixed, the quantity of the mixture is equal to the quantity of the ingredients ; the cost of the mixture is equal to the cost of the ingredients.

Ex. A wine-merchant has two kinds of wine which cost 72 cents and 40 cents a quart respectively. How much of each must he take to make a mixture of 50 quarts worth 60 cents a quart ?

Let x = the required number of quarts worth 72 cents a quart,
and y = the required number of quarts worth 40 cents a quart.

Then, $72x$ = the cost in cents of the first kind,
 $40y$ = the cost in cents of the second kind of wine,
and 3000 = the cost in cents of the mixture.

$$\begin{aligned}\therefore x + y &= 50, \\ 72x + 40y &= 3000.\end{aligned}$$

From which equations the values of x and y may be found.

31. A grocer mixed tea that cost him 42 cents a pound with tea that cost him 54 cents a pound. He had 30 pounds of the mixture, and by selling it at the rate of 60 cents a pound, he gained as much as 10 pounds of the cheaper tea cost him. How many pounds of each did he put into the mixture?
32. A grocer mixes tea that cost him 90 cents a pound with tea that cost him 28 cents a pound. The cost of the mixture is \$61.20. He sells the mixture at 50 cents a pound, and gains \$3.80. How many pounds of each did he put into the mixture?
33. A farmer has 28 bushels of barley worth 84 cents a bushel. With his barley he wishes to mix rye worth \$1.08 a bushel, and wheat worth \$1.44 a bushel, so that the mixture may be 100 bushels, and be worth \$1.20 a bushel. How many bushels of rye and of wheat must he take?

NOTE V. It is to be remembered that if a person can do a piece of work in x days, *the part* of the work he can do in *one* day will be represented by $\frac{1}{x}$.

- Ex. A and B together can do a piece of work in 48 days ;
 A and C together can do it in 30 days ; B and C
 together can do it in $26\frac{2}{3}$ days. How long will it
 take each to do the work ?

Let x = the number of days it will take A alone to do the work,
 y = the number of days it will take B alone to do the work,
 and z = the number of days it will take C alone to do the work.

Then, $\frac{1}{x}$, $\frac{1}{y}$, $\frac{1}{z}$, respectively, will denote the part each can do
 in a day,

and $\frac{1}{x} + \frac{1}{y}$ will denote the part A and B together can do in a day ;

but $\frac{1}{48}$ will denote the part A and B together can do in a day.

Therefore, $\frac{1}{x} + \frac{1}{y} = \frac{1}{48}$ (1)

Likewise, $\frac{1}{x} + \frac{1}{z} = \frac{1}{30}$ (2)

and $\frac{1}{y} + \frac{1}{z} = \frac{1}{26\frac{1}{2}} = \frac{3}{80}$ (3)

Add (1), (2), and (3), $\frac{2}{x} + \frac{2}{y} + \frac{2}{z} = \frac{11}{120}$ (4)

Multiply (1) by 2, $\frac{2}{x} + \frac{2}{y} = \frac{1}{24}$ (5)

Subtract (5) from (4), $\frac{2}{z} = \frac{1}{20}$
 $\therefore z = 40.$

Subtract the double of (2) from (4), $\frac{2}{y} = \frac{1}{40}$
 $\therefore y = 80.$

Subtract the double of (3) from (4), $\frac{2}{x} = \frac{1}{60}$
 $\therefore x = 120.$

34. A and B together earn \$40 in 6 days; A and C together earn \$54 in 9 days; B and C together earn \$80 in 15 days. What does each earn a day?
35. A cistern has three pipes, A, B, and C. A and B will fill it in 1 hour and 10 minutes; A and C in 1 hour and 24 minutes; B and C in 2 hours and 20 minutes. How long will it take each to fill it?
36. A warehouse will hold 24 boxes and 20 bales; 6 boxes and 14 bales will fill half of it. How many of each alone will it hold?
37. Two workmen together complete some work in 20 days; but if the first had worked twice as fast, and the second half as fast, they would have finished it in 15 days. How long would it take each alone to do the work?
38. A purse holds 19 crowns and 6 guineas; 4 crowns and 5 guineas fill $\frac{1}{3}$ of it. How many of each alone will it hold?

39. A piece of work can be completed by A, B, and C together in 10 days; by A and B together in 12 days; by B and C, if B works 15 days and C 30 days. How long will it take each alone to do the work?
40. A cistern has three pipes, A, B, and C. A and B will fill it in a minutes; A and C in b minutes; B and C in c minutes. How long will it take each alone to fill it?

NOTE VI. In considering the *rate of increase or decrease* in quantities, it is usual to take 100 as a *common standard of reference*, so that the increase or decrease is calculated for every 100, and therefore called *per cent*.

It is to be observed that the representative of the number resulting after an increase has taken place is $100 + \text{increase per cent}$; and after a decrease, $100 - \text{decrease per cent}$.

Interest depends upon the *time* for which the money is lent, as well as upon the *rate* charged; the rate charged being the rate on the principal for *one year*. Hence,

$$\text{Simple interest} = \frac{\text{Principal} \times \text{Rate} \times \text{Time}}{100},$$

where Time means *number of years or fraction of a year*.

$$\text{Amount} = \text{Principal} + \text{Interest}.$$

In questions relating to stocks, 100 is taken as the representative of the *stock*, the *price* represents its market value, and the *per cent* represents the *interest* which the *stock* bears. Thus, if six per cent stocks are quoted at 108, the meaning is, that the price of \$100 of the stock is \$108, and that the interest derived from \$100 of the *stock* will be $\frac{6}{100}$ of \$100, that is, \$6 a year. The rate of interest on the *money invested* will be $\frac{6}{108}$ of 6 per cent.

41. A man has \$10,000 invested. For a part of this sum he receives 5 per cent interest, and for the rest 4 per cent; the income for his 5 per cent investment is \$50 more than from his 4 per cent. How much has he in each investment?

42. A sum of money, at simple interest, amounted in 6 years to \$26,000, and in 10 years to \$30,000. Find the sum and the rate of interest.
43. A sum of money, at simple interest, amounted in 10 months to \$26,250, and in 18 months to \$27,250. Find the sum and the rate of interest.
44. A sum of money, at simple interest, amounted in m years to a dollars, and in n years to b dollars. Find the sum and the rate of interest.
45. A sum of money, at simple interest, amounted in a months to c dollars, and in b months to d dollars. Find the sum and the rate of interest.
46. A person has a certain capital invested at a certain rate per cent. Another person has \$1000 more capital, and his capital invested at one per cent better than the first, and receives an income \$80 greater. A third person has \$1500 more capital, and his capital invested at two per cent better than the first, and receives an income \$150 greater. Find the capital of each, and the rate at which it is invested.
47. A person has \$12,750 to invest. He can buy three per cent bonds at 81, and five per cents at 120. Find the amount of money he must invest in each in order to have the same income from each investment.
48. A and B each invested \$1500 in bonds; A in three per cents and B in four per cents. The bonds were bought at such prices that B received \$5 interest more than A. After both classes of bonds rose 10 points, they sold out, and A received \$50 more than B. What price was paid for each class of bonds?

49. A person invests \$10,000 in three per cent bonds, \$16,500 in three and one-half per cents, and has an income from both investments of \$1056.25. If his investments had been \$2750 more in the three per cents, and less in the three and one-half per cents, his income would have been $62\frac{1}{2}$ cents greater. What price was paid for each class of bonds?
50. The sum of \$2500 was divided into two unequal parts and invested, the smaller part at two per cent more than the larger. The *rate* of interest on the larger sum was afterwards increased by 1, and that of the smaller sum diminished by 1; and thus the *interest* of the whole was increased by one-fourth of its value. If the interest of the larger sum had been so increased, and no change had been made in the interest of the smaller sum, the interest of the whole would have been increased one-third of its value. Find the sums invested, and the rate per cent of each.

NOTE VII. If x represents the number of linear units in the length, and y in the width, of a rectangle, xy will represent the number of its units of surface; the surface unit having the same name as the linear unit of its sides.

51. If the sides of a rectangular field were each increased by 2 yards, the area would be increased by 220 square yards; if the length were increased and the breadth were diminished each by 5 yards, the area would be diminished by 185 square yards. What is its area?
52. If a given rectangular floor had been 3 feet longer and 2 feet broader it would have contained 64 square feet more; but if it had been 2 feet longer and 3 feet broader it would have contained 68 square feet more. Find the length and breadth of the floor.

53. In a certain rectangular garden there is a strawberry-bed whose sides are one-third of the lengths of the corresponding sides of the garden. The perimeter of the garden exceeds that of the bed by 200 yards; and if the greater side of the garden be increased by 3, and the other by 5 yards, the garden will be enlarged by 645 square yards. Find the length and breadth of the garden.

NOTE VIII. Care must be taken to express the conditions of a problem with reference to the same principal unit.

- Ex. In a mile race A gives B a start of 20 yards and beats him by 30 seconds. At the second trial A gives B a start of 32 seconds and beats him by $9\frac{2}{11}$ yards. Find the rate per hour at which each runs.

Let x = the number of yards A runs a second,
and y = the number of yards B runs a second.

Since there are 1760 yards in a mile,

$\frac{1760}{x}$ = the number of seconds it takes A to run a mile,

$\frac{1740}{y}$ and $\frac{1750\frac{6}{11}}{y}$ = the number of seconds B was running in the first and second trials,

Hence, $\frac{1740}{y} - \frac{1760}{x} = 30$, respectively.

and $\frac{1750\frac{6}{11}}{y} - \frac{1760}{x} = 32$.

The solution of these equations gives $x = 51\frac{3}{11}$ and $y = 51\frac{2}{11}$.

That is, A runs $\frac{51\frac{3}{11}}{1760}$, or $\frac{1}{300}$, of a mile in one second;

and in one hour, or 3600 seconds, runs 12 miles.

Likewise, B runs $10\frac{9}{11}$ miles in one hour.

54. In a mile race A gives B a start of 100 yards and beats him by 15 seconds. In the second trial A gives B a start of 45 seconds and is beaten by 22 yards. Find the rate of each in miles per hour.

55. In a mile race A gives B a start of 44 yards and beats him by 51 seconds. In the second trial A gives B a start of 1 minute and 15 seconds and is beaten by 88 yards. Find the rate of each in miles per hour.
56. The time which an express train takes to go 120 miles is $\frac{2}{3}$ of the time taken by an accommodation-train. The slower train loses as much time in stopping at different stations as it would take to travel 20 miles without stopping; the express-train loses only half as much time by stopping as the accommodation-train, and travels 15 miles an hour faster. Find the rate of each train in miles per hour.
57. A train moves from P towards Q, and an hour later a second train starts from Q and moves towards P at a rate of 10 miles an hour more than the first train; the trains meet half-way between P and Q. If the train from P had started an hour after the train from Q its rate must have been increased by 28 miles in order that the trains should meet at the half-way point. Find the distance from P to Q.
58. A passenger-train, after travelling an hour, meets with an accident which detains it one-half an hour; after which it proceeds at four-fifths of its usual rate, and arrives an hour and a quarter late. If the accident had happened 30 miles farther on, the train would have been only an hour late. Determine the usual rate of the train.
59. A passenger-train, after travelling an hour, is detained 15 minutes; after which it proceeds at three-fourths of its former rate, and arrives 24 minutes late. If the detention had taken place 5 miles farther on, the train would have been only 21 minutes late. Determine the usual rate of the train.

60. A man bought 10 oxen, 120 sheep, and 46 lambs. The cost of 3 sheep was equal to that of 5 lambs; an ox, a sheep, and a lamb together cost a number of dollars less by 57 than the whole number of animals bought; and the whole sum spent was \$2341.50. Find the price of an ox, a sheep, and a lamb, respectively.
61. A farmer sold 100 head of stock, consisting of horses, oxen, and sheep, so that the whole realized \$11.75 a head; while a horse, an ox, and a sheep were sold for \$110, \$62.50, and \$7.50, respectively. Had he sold one-fourth of the number of oxen that he did, and 25 more sheep, he would have received the same sum. Find the number of horses, oxen, and sheep, respectively, which were sold.
62. A, B, and C together subscribed \$100. If A's subscription had been one-tenth less, and B's one-tenth more, C's must have been increased by \$2 to make up the sum; but if A's had been one-eighth more, and B's one-eighth less, C's subscription would have been \$17.50. What did each subscribe?
63. A gives to B and C as much as each of them has; B gives to A and C as much as each of them then has; and C gives to A and B as much as each of them then has. In the end each of them has \$6. How much had each at first?
64. A pays to B and C as much as each of them has; B pays to A and C one-half as much as each of them then has; and C pays to A and B one-third of what each of them then has. In the end A finds that he has \$1.50, B \$4.16 $\frac{2}{3}$, C \$0.58 $\frac{1}{3}$. How much had each at first?

CHAPTER XIII.

INVOLUTION AND EVOLUTION.

198. THE operation of raising an expression to any required *power* is called **Involution**.

Every case of involution is merely an example of *multiplication*, in which the factors are *equal*. Thus,

$$(2a^3)^2 = 2a^3 \times 2a^3 = 4a^6.$$

199. A power of a simple expression is found by multiplying the exponent of each factor by the exponent of the required power, and taking the product of the resulting factors. The proof of the law of exponents, in its general form, is :

$$\begin{aligned} (a^m)^n &= a^m \times a^m \times a^m \times \dots \text{ to } n \text{ factors,} \\ &= a^{m+m+m+\dots \text{ to } n \text{ terms,}} \\ &= a^{mn}. \end{aligned}$$

Hence, if the exponent of the required power be a composite number, it may be resolved into prime factors, the power denoted by one of these factors may be found, and the result raised to a power denoted by another, and so on. Thus, the fourth power may be obtained by taking the second power of the second power; the sixth by taking the second power of the third power; the eighth by taking the second power of the second power of the second power.

200. From the **Law of Signs** in multiplication it is evident that,

I. All *even* powers of a number are *positive*.

II. All *odd* powers of a number have the *same sign* as the number itself.

Hence, no *even* power of *any* number can be *negative*; and of two compound expressions whose terms are identical but have opposite signs, the even powers are the same. Thus,

$$(b-a)^2 = \{-(a-b)\}^2 = (a-b)^2.$$

201. A method has been given, § 83, of finding, without actual multiplication, the powers of binomials which have the form $(a \pm b)$.

The same method may be employed when the terms of a binomial have *coefficients* or *exponents*.

$$(1) \quad (a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3.$$

$$\begin{aligned} (2) \quad (5x^2 - 2y^3)^3, \\ = (5x^2)^3 - 3(5x^2)^2(2y^3) + 3(5x^2)(2y^3)^2 - (2y^3)^3, \\ = 125x^6 - 150x^4y^3 + 60x^2y^6 - 8y^9. \end{aligned}$$

$$(3) \quad (a-b)^4 = a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4.$$

$$\begin{aligned} (4) \quad (x^2 - \tfrac{1}{2}y)^4, \\ = (x^2)^4 - 4(x^2)^3(\tfrac{1}{2}y) + 6(x^2)^2(\tfrac{1}{2}y)^2 - 4x^2(\tfrac{1}{2}y)^3 + (\tfrac{1}{2}y)^4, \\ = x^8 - 2x^6y + \tfrac{3}{2}x^4y^2 - \tfrac{1}{2}x^2y^3 + \tfrac{1}{16}y^4. \end{aligned}$$

202. In like manner, a *polynomial* of three or more terms may be raised to any power by enclosing its terms in parentheses, so as to give the expression the form of a binomial. Thus,

$$\begin{aligned} (1) \quad (a+b+c)^3 &= \{a+(b+c)\}^3, \\ &= a^3 + 3a^2(b+c) + 3a(b+c)^2 + (b+c)^3, \\ &= a^3 + 3a^2b + 3a^2c + 3ab^2 + 6abc \\ &\quad + 3ac^2 + b^3 + 3b^2c + 3bc^2 + c^3. \end{aligned}$$

$$\begin{aligned}
 (2) \quad & (x^3 - 2x^2 + 3x + 4)^2, \\
 & = \{ (x^3 - 2x^2) + (3x + 4) \}^2, \\
 & = (x^3 - 2x^2)^2 + 2(x^3 - 2x^2)(3x + 4) + (3x + 4)^2, \\
 & = x^6 - 4x^5 + 4x^4 + 6x^4 - 4x^3 - 16x^2 + 9x^2 + 24x + 16, \\
 & = x^6 - 4x^5 + 10x^4 - 4x^3 - 7x^2 + 24x + 16.
 \end{aligned}$$

EXERCISE 76.

Write the second members of the following equations :

1. $(a^3)^2 =$	11. $(2a^2bc^3)^4 =$	21. $(-3a^2b^2c)^5 =$
2. $(x^5)^3 =$	12. $(-5ax^3y^2)^3 =$	22. $(-3xy^3)^6 =$
3. $(x^2y^3)^2 =$	13. $(-7m^3nx^2y^4)^2 =$	23. $(-5a^2bx^3)^5 =$
4. $\left(\frac{a^3b^2}{2}\right)^4 =$	14. $\left(-\frac{2x^3y}{3abc}\right)^5 =$	24. $\left(-\frac{3ab^2}{4c^3}\right)^4 =$
5. $\left(\frac{3x^2y}{2a^3b^2}\right)^5 =$	15. $(3x + 1)^4 =$	25. $\left(-\frac{x^2y^3z^4}{2}\right)^7 =$
6. $(x + 2)^3 =$	16. $(2x - a)^4 =$	26. $(1 - a - a^2)^2 =$
7. $(x - 2)^4 =$	17. $(3x + 2a)^5 =$	27. $(2 - 3x + 4x^2)^3 =$
8. $(x + 3)^5 =$	18. $(2x - y)^4 =$	28. $(1 - 2x + x^2)^3 =$
9. $(1 + 2x)^5 =$	19. $(x^2y - 2xy^2)^6 =$	29. $(1 - x + x^2)^3 =$
10. $(2m - 1)^3 =$	20. $(ab - 3)^7 =$	30. $(1 + x + x^2)^4 =$

EVOLUTION.

203. The operation of finding any required *root* of an expression is called **Evolution**.

Every case of evolution is merely an example of *factoring*, in which the required factors are all *equal*. Thus, the square, cube, fourth..... roots of an expression are found by taking one of the *two, three, four..... equal factors* of the expression.

204. The symbol which denotes that a square root is to be extracted is $\sqrt{}$; and for other roots the same symbol is used, but with a figure written above to indicate the root, thus, $\sqrt[3]{}$, $\sqrt[4]{}$, etc., signifies the *third* root, *fourth* root, etc.

205. Since the cube of $a^3 = a^6$, the cube root of $a^6 = a^2$.

Since the *fourth* power of $2a^2 = 2^4a^8$, the *fourth* root of $2^4a^8 = 2a^2$.

Since the square of $abc = a^2b^2c^2$, the *square* root of $a^2b^2c^2 = abc$.

Since the *square* of $\frac{ab}{xy} = \frac{a^2b^2}{x^2y^2}$, the *square* root of $\frac{a^2b^2}{x^2y^2} = \frac{ab}{xy}$.

Hence, the root of a simple expression is found by *dividing the exponent of each factor by the index of the root, and taking the product of the resulting factors.*

206. It is evident from § 200 that

I. Any *even* root of a *positive* number will have the double sign, \pm .

II. There can be no *even* root of a *negative* number.

III. Any *odd* root of a number will have the same sign as the number.

$$\text{Thus, } \sqrt{\frac{16x^3}{81y^3}} = \pm \frac{4x}{9y}; \quad \sqrt[3]{-27m^3n^3} = -3mn^3;$$

$$\sqrt[4]{\frac{16x^8y^{12}}{81a^{16}}} = \pm \frac{2x^2y^3}{3a^4}.$$

But $\sqrt{-x^2}$ is neither $+x$ nor $-x$, for $(+x)^2 = +x^2$, and $(-x)^2 = +x^2$.

The indicated even root of a negative number is called an *impossible*, or *imaginary*, number.

207. If the root of a number expressed in figures is not readily detected, it may be found by resolving the number into its prime factors. Thus, to find the square root of 3,415,104 :

$$\begin{array}{r}
 2^3 \overline{) 3415104} \\
 2^3 \overline{) 426888} \\
 3^3 \overline{) 53361} \\
 7 \overline{) 5929} \\
 7 \overline{) 847} \\
 11 \overline{) 121} \\
 \hline
 11
 \end{array}$$

$$\therefore 3,415,104 = 2^8 \times 3^2 \times 7^2 \times 11^2.$$

$$\therefore \sqrt{3,415,104} = 2^3 \times 3 \times 7 \times 11 = 1848.$$

Simplify:

EXERCISE 77.

- $\sqrt{a^4}, \sqrt[4]{x^8}, \sqrt{4a^6b^2}, \sqrt[3]{64}, \sqrt[5]{a^5x^{10}y^{15}}, \sqrt[4]{16a^{12}b^4c^8}, \sqrt[5]{-32a^{15}}.$
- $\sqrt[3]{-1728c^6a^{12}x^3y^9}, \sqrt[3]{3375b^{21}z^{15}}, \sqrt[4]{3111696c^{16}z^4}.$
- $\sqrt{53361b^4c^8y^{12}z^{16}}, \sqrt[3]{-\frac{216b^3c^{15}}{343z^{24}}}, \sqrt[6]{\frac{64x^{18}}{729z^{30}}}.$
- $\sqrt{25a^2b^4c^2} + \sqrt[3]{8a^3b^6c^3} - \sqrt[4]{81a^4b^8c^4} - \sqrt[5]{32a^5b^{10}c^5}.$
- $\sqrt[3]{27x^3y^3} \times \sqrt[5]{243y^5z^5} \times \sqrt{16x^4z^2}.$

When $a = 1, b = 3, x = 2, y = 6$, find the values of :

- $4\sqrt{2x} - \sqrt{abxy} + 5\sqrt{a^2b^2xy}.$
- $2a\sqrt{8ax} + b\sqrt[3]{12by} + 4abx\sqrt{bxy}.$
- $\sqrt{a^2 + 2ab + b^2} \times \sqrt[3]{a^3 + 3a^2b + 3ab^2 + b^3}.$
- $\sqrt[3]{b^3 - 3b^2a + 3ba^2 - a^3} \div \sqrt{b^2 + a^2 - 2ab}.$

SQUARE ROOTS OF COMPOUND EXPRESSIONS.

208. Since the square of $a + b$ is $a^2 + 2ab + b^2$, the square root of $a^2 + 2ab + b^2$ is $a + b$.

It is required to find a method of extracting the root $a + b$ when $a^2 + 2ab + b^2$ is given :

Ex. The first term, a , of the root is obviously the square root of the first term, a^2 , in the expression.

$$\begin{array}{r} a^2 + 2ab + b^2 \overline{) a + b} \\ \underline{a^2} \\ 2a + \overline{b} \end{array}$$
 If the a^2 is subtracted from the given expression, the remainder is $2ab + b^2$. Therefore the second term, b , of the root is obtained when the first term of this remainder is divided by $2a$, that is, by *double the part of*

the root already found. Also, since $2ab + b^2 = (2a + b)b$, the divisor is completed by adding to the trial-divisor the new term of the root.

Ex. Find the square root of $25x^2 - 20x^3y + 4x^4y^2$.

$$\begin{array}{r} 25x^2 - 20x^3y + 4x^4y^2 \overline{) 5x - 2x^2y} \\ \underline{25x^2} \\ 10x - 2x^2y \end{array}$$

$$\begin{array}{r} \overline{) -20x^3y + 4x^4y^2} \\ \underline{-20x^3y + 4x^4y^2} \end{array}$$

The expression is *arranged* according to the ascending powers of x .

The square root of the first term is $5x$, and $5x$ is placed at the right of the given expression, for the first term of the root.

The second term of the root, $-2x^2y$, is obtained by dividing $-20x^3y$ by $10x$, and this new term of the root is also annexed to the divisor, $10x$, to complete the divisor.

209. The same method will apply to longer expressions, if care be taken to obtain the *trial-divisor* at each stage of the process, by *doubling the part of the root already found*; and to obtain the *complete divisor* by *annexing the new term of the root to the trial-divisor*.

Ex. Find the square root of

$$\begin{array}{r}
 1 + 10x^2 + 25x^4 + 16x^6 - 24x^5 - 20x^3 - 4x. \\
 16x^6 - 24x^5 + 25x^4 - 20x^3 + 10x^2 - 4x + 1 \quad \underline{4x^3 - 3x^2 + 2x - 1} \\
 16x^6 \\
 8x^3 - 3x^2 \quad \underline{-24x^5 + 25x^4} \\
 \quad \quad \quad \underline{-24x^5 + 9x^4} \\
 8x^3 - 6x^2 + 2x \quad \underline{16x^4 - 20x^3 + 10x^2} \\
 \quad \quad \quad \underline{16x^4 - 12x^3 + 4x^2} \\
 8x^3 - 6x^2 + 4x - 1 \quad \underline{-8x^3 + 6x^2 - 4x + 1} \\
 \quad \quad \quad \underline{-8x^3 + 6x^2 - 4x + 1}
 \end{array}$$

The expression is arranged according to the descending powers of x .

It will be noticed that each successive trial-divisor may be obtained by taking the preceding complete divisor with its *last term doubled*.

EXERCISE 78.

Extract the square roots of:

- $a^4 + 4a^3 + 2a - 4a + 1.$
- $x^4 - 2x^3y + 3x^2y^2 - 2xy^3 + y^4.$
- $4a^6 - 12a^5x + 5a^4x^2 + 6a^3x^3 + a^2x^4.$
- $9x^6 - 12x^3y^3 + 16x^2y^4 - 24x^4y^2 + 4y^6 + 16xy^5.$
- $4a^8 + 16c^8 + 16a^6c^2 - 32a^2c^6.$
- $4x^4 + 9 - 30x - 20x^3 + 37x^2.$
- $16x^4 - 16abx^2 + 16b^2x^2 + 4a^2b^2 - 8ab^3 + 4b^4.$
- $x^6 + 25x^2 + 10x^4 - 4x^5 - 20x^3 + 16 - 24x.$
- $x^6 + 8x^4y^2 - 4x^5y - 4xy^5 + 8x^2y^4 - 10x^3y^3 + y^6.$
- $4 - 12a - 11a^4 + 5a^2 - 4a^5 + 4a^6 + 14a^3.$
- $9a^2 - 6ab + 30ac + 6ad + b^2 - 10bc - 2bd$
 $\quad \quad \quad + 25c^2 + 10cd + d^2.$

$$12. 25x^6 - 31x^4y^2 + 34x^2y^4 - 30xy^6 + y^8 - 8xy^5 + 10x^2y^4.$$

$$13. m^8 - 4m^7 + 10m^6 - 20m^5 - 44m^4 + 35m^3 + 46m^2 - 40m + 25.$$

$$14. x^4 - x^3y - \frac{7}{4}x^2y^2 + xy^3 + y^4.$$

$$15. x^4 - 4x^3y + 6x^2y^2 - 6xy^3 + 5y^4 - \frac{2y^5}{x} + \frac{y^6}{x^2}.$$

$$16. \frac{a^4}{9} - \frac{a^3x}{2} + \frac{43}{48}a^2x^2 - \frac{3}{4}ax^3 + \frac{x^4}{4}.$$

$$17. 1 + \frac{4}{x} + \frac{10}{x^2} + \frac{20}{x^3} + \frac{25}{x^4} + \frac{24}{x^5} + \frac{16}{x^6}.$$

$$18. \frac{a^2}{b^2} - \frac{2a}{b} + 3 - \frac{2b}{a} + \frac{b^2}{a^2}. \quad 19. x^4 + x^3 - \frac{5x^2}{12} - \frac{x}{3} + \frac{1}{9}.$$

SQUARE ROOTS OF ARITHMETICAL NUMBERS.

210. In the general method of extracting the square root of a number expressed by figures, the first step is to mark off the figures into *groups*.

Since $1 = 1^2$, $100 = 10^2$, $10,000 = 100^2$, and so on, it is evident that the square root of any number between 1 and 100 lies between 1 and 10; the square root of any number between 100 and 10,000 lies between 10 and 100. In other words, the square root of any integral number expressed by *one* or *two* figures is a number of *one* figure; the square root of any integral number expressed by *three* or *four* figures is a number of *two* figures; and so on.

If, therefore, an integral number be divided into groups of two figures each, from right to left, the number of figures in the root will be equal to the number of groups of figures. The last group to the left may have one or two figures.

Ex. Find the square root of 3249.

$$\begin{array}{r} 32\ 49\ (57 \\ 25 \\ \hline 107) 7\ 49 \\ \underline{7\ 49} \end{array}$$

In this case, a in the typical form $a^2 + 2ab + b^2$ represents 5 *tens*, or 50, and b represents 7 ones. The 25 subtracted is really 2500, that is, a^2 , and the complete divisor, $2a + b$, is $2 \times 50 + 7$, or 107.

211. The same method will apply to numbers of more than two groups by considering *a* in the typical form to represent at each step *the part of the root already found*.

It must be observed that *a* represents so many tens with respect to the next figure of the root.

Ex. Find the square root of 5,322,249.

$$\begin{array}{r}
 5\ 32\ 22\ 49\ (2307 \\
 \quad 4 \\
 43\overline{)132} \\
 \quad 129 \\
 \hline
 4607\overline{)32249} \\
 \quad 32249 \\
 \hline
 \end{array}$$

212. If the square root of a number have decimal places, the number itself will have *twice* as many.

Thus, if 0.21 be the square root of some number, this number will be $(0.21)^2 = 0.21 \times 0.21 = 0.0441$; and if 0.111 be the root, the number will be $(0.111)^2 = 0.111 \times 0.111 = 0.012321$.

Therefore, the number of *decimal* places in every square decimal will be *even*, and the number of decimal places in the root will be *half* as many as in the given number itself.

Hence, if the given square number contains a decimal, we divide it into groups of two figures each, by beginning at the decimal point and marking toward the left for the integral number, and toward the right for the decimal. We must be careful to have the last group on the right of the decimal point contain *two* figures, annexing a cipher when necessary.

Ex. Find the square roots of 41.2164 and 965.9664.

$$\begin{array}{r}
 41.21\ 64\ (6.42 \\
 \quad 36 \\
 124\overline{)521} \\
 \quad 496 \\
 \hline
 1282\overline{)2564} \\
 \quad 2564 \\
 \hline
 \end{array}$$

$$\begin{array}{r}
 9\ 65.96\ 64\ (31.08 \\
 \quad 9 \\
 61\overline{)65} \\
 \quad 61 \\
 \hline
 6208\overline{)49664} \\
 \quad 49664 \\
 \hline
 \end{array}$$

It is seen from the grouping that the root of the first example will have one integral and two decimal places, and that the root of the second example will have two integral and two decimal places.

213. If a number contains an *odd* number of decimal places, or if any number gives a *remainder* when as many figures in the root have been obtained as the given number has groups, then its exact square root cannot be found. We may, however, approximate to its exact root as near as we please by annexing ciphers and continuing the operation.

Ex. Find the square roots of 3 and 357.357.

$$\begin{array}{r}
 3. (1.732..... \\
 1 \\
 27 \overline{)200} \\
 \underline{189} \\
 343 \overline{)1100} \\
 \underline{1029} \\
 3462 \overline{)7100} \\
 \underline{6924}
 \end{array}$$

$$\begin{array}{r}
 357.3570 (18.903..... \\
 1 \\
 28 \overline{)257} \\
 \underline{224} \\
 369 \overline{)3335} \\
 \underline{3321} \\
 37803 \overline{)147000} \\
 \underline{113409}
 \end{array}$$

EXERCISE 79.

Extract the square roots of:

- 120,409; 4816.36; 1867.1041; 1435.6521; 64.128064.
- 16,803.9369; 4.54499761; 0.24373969; 0.5687573056.
- 0.9; 6.21; 0.43; 0.00852; 17; 129; 347.259.
- 14,295.387; 2.5; 2000; 0.3; 0.03; 111.
- 0.00111; 0.004; 0.005; 2; 5; 3.25; 8.6.
- $\frac{1}{4}$; $\frac{1}{16}$; $1\frac{1}{4}$; $1\frac{1}{16}$; $3\frac{3}{4}$; $3\frac{3}{16}$.
- $\frac{1}{2}$; $\frac{3}{4}$; $\frac{5}{8}$; $1\frac{1}{2}$; $1\frac{1}{4}$; $1\frac{1}{8}$.

CUBE ROOTS OF COMPOUND EXPRESSIONS.

214. Since the cube of $a+b$ is $a^3 + 3a^2b + 3ab^2 + b^3$, the cube root of $a^3 + 3a^2b + 3ab^2 + b^3$ is $a+b$.

It is required to find a method for extracting the cube root $a+b$ when $a^3 + 3a^2b + 3ab^2 + b^3$ is given:

(1) Find the cube root of $a^3 + 3a^2b + 3ab^2 + b^3$.

$$\begin{array}{r}
 a^3 + 3a^2b + 3ab^2 + b^3 \big| a + b \\
 3a^2 a^3 \\
 \hline
 + 3ab + b^2 3a^2b + 3ab^2 + b^3 \\
 3a^2 + 3ab + b^2 3a^2b + 3ab^2 + b^3 \\
 \hline
 3a^2b + 3ab^2 + b^3
 \end{array}$$

The first term a of the root is obviously the cube root of the first term a^3 of the given expression.

If a^3 is subtracted, the remainder is $3a^2b + 3ab^2 + b^3$; therefore, the second term b of the root is obtained by dividing the first term of this remainder by *three times the square of a* .

Also, since $3a^2b + 3ab^2 + b^3 = (3a^2 + 3ab + b^2)b$, the *complete divisor* is obtained by adding $3ab + b^2$ to the *trial divisor* $3a^2$.

(2) Find the cube root of $8x^3 + 36x^2y + 54xy^2 + 27y^3$.

$$\begin{array}{r}
 8x^3 + 36x^2y + 54xy^2 + 27y^3 \big| 2x + 3y \\
 12x^2 8x^3 \\
 \hline
 (6x+3y)3y = 36x^2y + 54xy^2 + 27y^3 \\
 12x^2 + 18xy + 9y^2 36x^2y + 54xy^2 + 27y^3 \\
 \hline
 36x^2y + 54xy^2 + 27y^3
 \end{array}$$

The cube root of the first term is $2x$, and this is therefore the first term of the root.

The second term of the root, $3y$, is obtained by dividing $36x^2y$ by $3(2x)^2 = 12x^2$, which corresponds to $3a^2$ in the typical form, and is completed by annexing to $12x^2$ the expression $\{3(2x) + 3y\}3y = 18xy + 9y^2$, which corresponds to $3ab + b^2$, in the typical form.

215. The same method may be applied to longer expressions by considering a in the typical form $3a^2 + 3ab + b^2$ to represent at each stage of the process *the part of the root already found*.

Thus, if the part of the root already found be $x + y$, then $3a^2$ of the typical form will be represented by $3(x + y)^2$; and if the third term of the root be $+z$, the $3ab + b^2$ will be represented by $3(x + y)z + z^2$. So that the complete divisor, $3a^2 + 3ab + b^2$, will be represented by $3(x + y)^2 + 3(x + y)z + z^2$.

Ex. Find the cube root of $x^6 - 3x^5 + 5x^3 - 3x - 1$.

$$\begin{array}{r}
 \phantom{\frac{3x^4}{3x^4-3x^3+x^2}} \phantom{\frac{x^6}{-3x^5+5x^3}} \phantom{\frac{x^2-x-1}{-3x^4+6x^3-3x-1}} \\
 \phantom{\frac{3x^4}{3x^4-3x^3+x^2}} \phantom{\frac{x^6}{-3x^5+5x^3}} \phantom{\frac{x^2-x-1}{-3x^4+6x^3-3x-1}} \\
 (3x^2-x)(-x)= \frac{3x^4}{3x^4-3x^3+x^2} \phantom{\frac{x^6}{-3x^5+5x^3}} \phantom{\frac{x^2-x-1}{-3x^4+6x^3-3x-1}} \\
 \phantom{\frac{3x^4}{3x^4-3x^3+x^2}} \phantom{\frac{x^6}{-3x^5+5x^3}} \phantom{\frac{x^2-x-1}{-3x^4+6x^3-3x-1}} \\
 (3x^2-3x-1)(-1)= \frac{3x^4-6x^3+3x^2}{3x^4-6x^3} \phantom{\frac{x^6}{-3x^5+5x^3}} \phantom{\frac{x^2-x-1}{-3x^4+6x^3-3x-1}} \\
 \phantom{\frac{3x^4}{3x^4-3x^3+x^2}} \phantom{\frac{x^6}{-3x^5+5x^3}} \phantom{\frac{x^2-x-1}{-3x^4+6x^3-3x-1}}
 \end{array}$$

The root is placed above the given expression for convenience of arrangement.

The first term of the root, x^2 , is obtained by taking the cube root of the first term of the given expression; and the first trial-divisor, $3x^4$, is obtained by taking three times the square of this term of the root.

The first complete divisor is found by annexing to the trial-divisor $(3x^2-x)(-x)$, which expression corresponds to $(3a+b)b$ in the typical form.

The part of the root already found (a) is now represented by x^2-x ; therefore, $3a^2$ is represented by $3(x^2-x)^2 = 3x^4-6x^3+3x^2$, the second trial divisor; and $(3a+b)b$ by $(3x^2-3x-1)(-1)$; therefore, in the second complete divisor, $3a^2+(3a+b)b$ is represented by $(3x^4-6x^3+3x^2)+(-3x^2-3x-1) \times (-1) = 3x^4-6x^3+3x+1$.

EXERCISE 80.

Find the cube roots of:

- $x^3+6x^2y+12xy^2+8y^3$
- $a^3-9a^2+27a-27$
- $x^3+12x^2+48x+64$
- $x^6-3ax^5+5a^3x^3-3a^5x-a^6$
- $x^6+3x^5+6x^4+7x^3+6x^2+3x+1$
- $1-9x+39x^2-99x^3+156x^4-144x^5+64x^6$
- $a^6-6a^5+9a^4+4a^3-9a^2-6a-1$
- $64x^6+192x^5+144x^4-32x^3-36x^2+12x-1$
- $1-3x+6x^2-10x^3+12x^4-12x^5+10x^6-6x^7+3x^8-x^9$

10. $a^6 + 9a^5b - 135a^3b^3 + 729ab^5 - 729b^6$.
 11. $c^6 - 12bc^5 + 60b^2c^4 - 160b^3c^3 + 240b^4c^2 - 192b^5c + 64b^6$.
 12. $8a^6 + 48a^5b + 60a^4b^2 - 80a^3b^3 - 90a^2b^4 + 108ab^5 - 27b^6$.

CUBE ROOTS OF ARITHMETICAL NUMBERS.

216. In extracting the cube root of a number expressed by figures, the first step is to mark it off into groups.

Since $1 = 1^3$, $1000 = 10^3$, $1,000,000 = 100^3$, and so on, it follows that the cube root of any integral number between 1 and 1000, that is, of any integral number which has *one, two, or three* figures, is a number of *one* figure; and that the cube root of any integral number between 1000 and 1,000,000, that is, of any integral number which has *four, five, or six* figures, is a number of *two* figures; and so on.

If, therefore, an integral number be divided into groups of three figures each, from right to left, the number of figures in the root will be equal to the number of groups. The last group to the left may have one, two, or three figures.

217. If the cube root of a number contain any decimal figures, the number itself will contain *three times* as many.

Thus, if 0.3 is the cube root of a number, the number is

$$0.3 \times 0.3 \times 0.3 = 0.027.$$

Hence, if the given cube number have decimal places, we divide it into groups of three figures each, by beginning at the decimal point and marking toward the left for the integral number, and toward the right for the decimal. We must be careful to have the last group on the right of the decimal point contain three figures, annexing ciphers when necessary.

If the given number be not a perfect cube, ciphers may be annexed, and a value of the root may be found as near to the *true* value as we please.

218. It is to be observed that if a denotes the first term of the root, and b the second term, the *first complete divisor* is

$$3a^2 + 3ab + b^2,$$

and the *second trial-divisor* is $3(a + b)^2$, that is,

$$3a^2 + 6ab + 3b^2,$$

which may be obtained from the preceding complete divisor by adding to it its second term and twice its third term:

$$\begin{array}{r} 3a^2 + 3ab + b^2 \\ + 3ab + 2b^2 \\ \hline 3a^2 + 6ab + 3b^2 \end{array}$$

a method which will very much shorten the work in long arithmetical examples.

219. Ex. Extract the cube root of 5 to five places of decimals.

	5.000 (1.70997
	1
$3 \times 10^2 = 300$	4 000
$3(10 \times 7) = 210$	
$7^2 = 49$	
559	3 913
259	87 000 000
$3 \times 1700^2 = 8670000$	
$3(1700 \times 9) = 45900$	
$9^2 = 81$	
8715981	78 443 829
45981	8 556 171 0
$3 \times 1709^2 = 8762043$	7 885 838 7
	670 332 30
	613 343 01

After the first two figures of the root are found, the next trial-divisor is obtained by bringing down the sum of the 210 and 49 obtained in completing the preceding divisor; then adding the three lines connected by the brace, and annexing two ciphers to the result.

The last two figures of the root are found by division. The rule in such cases is, that two less than the number of figures already obtained may be found without error by division, the divisor to be employed being three times the square of the part of the root already found.

EXERCISE 81.

Find the cube roots of:

- | | | |
|----------------------|---------------------|--------------------|
| 1. 274,625. | 7. 1601.613. | 13. 33,076.161. |
| 2. 110,592. | 8. 1,259,712. | 14. 102,503.232. |
| 3. 262,144. | 9. 2.803221. | 15. 820.025856. |
| 4. 884.736. | 10. 7,077,888. | 16. 8653.002877. |
| 5. 109,215,352. | 11. 12.812904. | 17. 1.371330631. |
| 6. 1,481,544. | 12. 56.623104. | 18. 20,910.518875. |
| 19. 91.398648466125. | 20. 5.340104393239. | |
21. Find to four figures the cube roots of 2.5; 0.2; 0.01; 4; 0.4.

220. Since the fourth power is the square of the square, and the sixth power the square of the cube; the *fourth root* is the *square root* of the *square root*, and the *sixth root* is the *cube root* of the *square root*. In like manner, the eighth, ninth, twelfth..... roots may be found.

EXERCISE 82.

Find the fourth roots of:

- $81a^4 - 540a^3b + 1350a^2b^2 - 1500ab^3 + 625b^4$.
- $1 - 4x + 10x^2 - 16x^3 + 19x^4 - 16x^5 + 10x^6 - 4x^7 + x^8$.

Find the sixth roots of:

- $64 - 192x + 240x^2 - 160x^3 + 60x^4 - 12x^5 + x^6$.
- $729x^6 - 1458x^5 + 1215x^4 - 540x^3 + 135x^2 - 18x + 1$.

Find the eighth root of:

- $1 - 8y + 28y^2 - 56y^3 + 70y^4 - 56y^5 + 28y^6 - 8y^7 + y^8$.

CHAPTER XIV.

QUADRATIC EQUATIONS.

221. An equation which contains the *square* of the unknown quantity, but no higher power, is called a **quadratic equation**.

222. If the equation contains the *square only*, it is called a **pure quadratic**; but if it contains the *first power also*, it is called an **affected quadratic**.

PURE QUADRATIC EQUATIONS.

(1) Solve the equation $5x^2 - 48 = 2x^2$.

$5x^2 - 48 = 2x^2$	It will be observed that there are <i>two</i> roots of
$3x^2 = 48$	equal value but of opposite signs; and there are
$x^2 = 16$	only two, for if the square root of the equation,
$\therefore x = \pm 4$	$x^2 = 16$, were written $\pm x = \pm 4$, there would be
	only two values of x ; since the equation $-x$
	$= +4$ gives $x = -4$, and the equation $-x = -4$ gives $x = 4$.

Hence, to solve a pure quadratic,

Collect the unknown numbers on one side, and the known numbers on the other; divide by the coefficient of the unknown number; and extract the square root of each side of the resulting equation.

(2) Solve the equation $3x^2 - 15 = 0$.

$3x^2 - 15 = 0$	It will be observed that the square root of 5
$3x^2 = 15$	cannot be found exactly, but an approximate
$x^2 = 5$	value of it to any assigned degree of accuracy
$\therefore x = \pm \sqrt{5}$	may be found.

223. A root which is indicated, but which can be found only approximately, is called a **Surd**.

Ex. Solve the equation $3x^2 + 15 = 0$.

$$3x^2 + 15 = 0$$

$$3x^2 = -15$$

$$x^2 = -5$$

$$\therefore x = \pm \sqrt{-5}$$

It will be observed that the square root of -5 cannot be found even approximately; for the square of any number, positive or negative, is positive.

224. A root which is indicated, but which cannot be found exactly or approximately, is **imaginary**. § 206.

EXERCISE 83.

Solve:

1. $x^2 - 3 = 46$.

6. $5x^2 - 9 = 2x^2 + 24$.

2. $2(x^2 - 1) - 3(x^2 + 1) + 14 = 0$.

7. $(x + 2)^2 = 4x + 5$.

3. $\frac{x^2 - 5}{3} + \frac{2x^2 + 1}{6} = \frac{1}{2}$.

8. $\frac{x^2}{5} - \frac{x^2 - 10}{15} = 7 - \frac{50 + x^2}{25}$.

4. $\frac{3}{1+x} + \frac{3}{1-x} = 8$.

9. $\frac{3x^2 - 27}{x^2 + 3} + \frac{90 + 4x^2}{x^2 + 9} = 7$.

5. $\frac{3}{4x^2} - \frac{1}{6x^2} = \frac{7}{3}$.

10. $8x + \frac{7}{x} = \frac{65x}{7}$.

11. $\frac{4x^2 + 5}{10} - \frac{2x^2 - 5}{15} = \frac{7x^2 - 25}{20}$.

12. $\frac{10x^2 + 17}{18} - \frac{12x^2 + 2}{11x^2 - 8} = \frac{5x^2 - 4}{9}$.

13. $\frac{14x^2 + 16}{21} - \frac{2x^2 + 8}{8x^2 - 11} = \frac{2x^2}{3}$.

14. $x^2 + bx + a = bx(1 - bx)$.

15. $mx^2 + n = q$.

16. $x^2 - ax + b = ax(x - 1)$.

AFFECTED QUADRATIC EQUATIONS.

225. Since $(ax \pm b)^2 = a^2x^2 \pm 2abx + b^2$, it is evident that the expression $a^2x^2 \pm 2abx$ lacks only the *third term, b^2* , of being a complete square.

It will be seen that this third term is *the square of the quotient obtained from dividing the second term by twice the square root of the first term.*

226. Every affected quadratic may be made to assume the form of $a^2x^2 \pm 2abx = c$.

The first step in the solution of such an equation is to *complete the square*; that is, to add to each side *the square of the quotient obtained from dividing the second term by twice the square root of the first term.*

The second step is to *extract the square root* of each side of the resulting equation.

The third and last step is to *reduce* the resulting simple equation.

(1) Solve the equation $16x^2 + 5x - 3 = 7x^2 - x + 45$.

$$16x^2 + 5x - 3 = 7x^2 - x + 45.$$

Simplify,

$$9x^2 + 6x = 48.$$

Complete the square, $9x^2 + 6x + 1 = 49$.

Extract the root,

$$3x + 1 = \pm 7.$$

Reduce,

$$3x = -1 + 7 \text{ or } -1 - 7,$$

$$3x = 6 \text{ or } -8.$$

$$\therefore x = 2 \text{ or } -2\frac{2}{3}.$$

Verify by substituting 2 for x in the equation

$$16x^2 + 5x - 3 = 7x^2 - x + 45,$$

$$16(2)^2 + 5(2) - 3 = 7(2)^2 - (2) + 45,$$

$$64 + 10 - 3 = 28 - 2 + 45,$$

$$71 = 71.$$

Verify by substituting $-2\frac{1}{3}$ for x in the equation

$$\begin{aligned} 16x^2 + 5x - 3 &= 7x^2 - x + 45, \\ 16\left(-\frac{2}{3}\right)^2 + 5\left(-\frac{2}{3}\right) - 3 &= 7\left(-\frac{2}{3}\right)^2 - \left(-\frac{2}{3}\right) + 45; \\ \frac{104}{9} - \frac{10}{3} - 3 &= \frac{28}{9} + \frac{2}{3} + 45, \\ 1024 - 120 - 27 &= 448 + 24 + 405, \\ 877 &= 877. \end{aligned}$$

(2) Solve the equation $3x^2 - 4x = 32$.

Since the exact root of 3, the coefficient of x^2 , cannot be found, it is necessary to multiply or divide each term of the equation by 3 to make the coefficient of x^2 a *square number*.

$$\begin{aligned} \text{Multiply by 3,} \quad & 9x^2 - 12x = 96. \\ \text{Complete the square,} \quad & 9x^2 - 12x + 4 = 100. \\ \text{Extract the root,} \quad & 3x - 2 = \pm 10. \\ \text{Reduce,} \quad & 3x = 2 + 10 \text{ or } 2 - 10; \\ & 3x = 12 \text{ or } -8. \\ \therefore x &= 4 \text{ or } -2\frac{2}{3}. \end{aligned}$$

$$\begin{aligned} \text{Or, divide by 3,} \quad & x^2 - \frac{4x}{3} = \frac{32}{3}. \\ \text{Complete the square,} \quad & x^2 - \frac{4x}{3} + \frac{4}{9} = \frac{32}{3} + \frac{4}{9} = \frac{100}{9}. \\ \text{Extract the root,} \quad & x - \frac{2}{3} = \pm \frac{10}{3}. \\ & \therefore x = \frac{2 \pm 10}{3}, \\ & = 4 \text{ or } -2\frac{2}{3}. \end{aligned}$$

Verify by substituting 4 for x in the original equation,

$$\begin{aligned} 48 - 16 &= 32, \\ 32 &= 32. \end{aligned}$$

Verify by substituting $-2\frac{2}{3}$ for x in the original equation,

$$\begin{aligned} 21\frac{1}{3} - (-10\frac{2}{3}) &= 32, \\ 32 &= 32. \end{aligned}$$

(3) Solve the equation $-3x^2 + 5x = -2$.

Since the *even* root of a *negative* number is impossible, it is necessary to change the sign of each term. The resulting equation is,

$$3x^2 - 5x = 2.$$

Multiply by 3,

$$9x^2 - 15x = 6.$$

Complete the square, $9x^2 - 15x + \frac{25}{4} = \frac{49}{4}$.

Extract the root, $3x - \frac{5}{2} = \pm \frac{7}{2}$.

Reduce, $3x = \frac{5 \pm 7}{2}$,

$$3x = 6 \text{ or } -1.$$

$$\therefore x = 2 \text{ or } -\frac{1}{3}.$$

Or, divide by 3,

$$x^2 - \frac{5x}{3} = \frac{2}{3}.$$

Complete the square, $x^2 - \frac{5x}{3} + \frac{25}{36} = \frac{49}{36}$.

Extract the root, $x - \frac{5}{6} = \pm \frac{7}{6}$.

$$\therefore x = \frac{5 \pm 7}{6},$$

$$= 2 \text{ or } -\frac{1}{3}.$$

If the equation $3x^2 - 5x = 2$ be multiplied by *four times the coefficient of x^2* , fractions will be avoided :

$$36x^2 - 60x = 24.$$

Complete the square, $36x^2 - 60x + 25 = 49$.

Extract the root, $6x - 5 = \pm 7$,

$$6x = 5 \pm 7,$$

$$6x = 12 \text{ or } -2.$$

$$\therefore x = 2 \text{ or } -\frac{1}{3}.$$

It will be observed that the number added to complete the square by this last method is *the square of the coefficient of x* in the original equation $3x^2 - 5x = 2$.

(4) Solve the equation $\frac{3}{5-x} - \frac{1}{2x-5} = 2$.

Simplify (as in simple equations),

$$4x^2 - 23x = -30.$$

Multiply by four times the coefficient of x^2 , and add to each side the square of the coefficient of x ,

$$64x^2 - () + (23)^2 = 529 - 480 = 49.$$

Extract the root,

$$8x - 23 = \pm 7.$$

Reduce,

$$8x = 23 \pm 7;$$

$$8x = 30 \text{ or } 16.$$

$$\therefore x = 3\frac{3}{4} \text{ or } 2.$$

If a trinomial is a perfect square, its square root is found by taking the square roots of the *first* and *third* terms and connecting them by the *sign* of the middle term. It is not necessary, therefore, in completing the square, to write the middle term, but its place may be indicated as in this example.

(5) Solve the equation $72x^2 - 30x = -7$.

Since $72 = 2^3 \times 3^2$, if the equation be multiplied by 2 the coefficient of x^2 in the resulting equation, $144x^2 - 60x = -14$, will be a square number, and the term required to complete the square will be $(\frac{5}{2})^2 = (\frac{5}{2})^2 = \frac{25}{4}$. Hence, if the original equation be multiplied by 4×2 , the coefficient of x^2 in the result will be a square number, and fractions will be avoided in the work.

Multiply the given equation by 8,

$$576x^2 - 240x = -56.$$

Complete the square, $576x^2 - () + 25 = -31$.

Extract the root,

$$24x - 5 = \pm \sqrt{-31}.$$

Reduce,

$$24x = 5 \pm \sqrt{-31}.$$

$$\therefore x = \frac{1}{24}(5 \pm \sqrt{-31}).$$

NOTE. In solving the following equations, care must be taken to select the method best adapted to the example under consideration.

Solve:

EXERCISE 84.

- | | | |
|------------------------------------|--------------------------|------------------------|
| 1. $x^2 + 4x = 12$. | 4. $x^2 - 7x = 8$. | 7. $x^2 - x = 6$. |
| 2. $x^2 - 6x = 16$. | 5. $3x^2 - 4x = 7$. | 8. $5x^2 - 3x = 2$. |
| 3. $x^2 - 12x + 6 = \frac{1}{4}$. | 6. $12x^2 + x - 1 = 0$. | 9. $2x^2 - 27x = 14$. |

$$10. x^2 - \frac{2x}{3} + \frac{1}{12} = 0.$$

$$13. \frac{x+1}{x+4} = \frac{2x-1}{x+6}.$$

$$11. \frac{x^2}{2} - \frac{x}{3} = 2(x+2).$$

$$14. \frac{x}{x+1} - \frac{x+3}{2(x+4)} = -\frac{1}{18}.$$

$$12. \frac{3x}{4} + \frac{4}{3x} = \frac{13}{6}.$$

$$15. \frac{2}{x-1} = \frac{3}{x-2} + \frac{2}{x-4}.$$

$$16. 5x(x-3) - 2(x^2-6) = (x+3)(x+4).$$

$$17. \frac{3x}{2(x+1)} - \frac{5}{8} = \frac{3x^2}{x^2-1} - \frac{23}{4(x-1)}.$$

$$18. (x-2)(x-4) - 2(x-1)(x-3) = 0.$$

$$19. \frac{1}{7}(x-4) - \frac{2}{5}(x-2) = \frac{1}{x}(2x+3).$$

$$20. \frac{2}{5}(3x^2-x-5) - \frac{1}{3}(x^2-1) = 2(x-2)^2.$$

$$21. \frac{2x}{15} + \frac{3x-50}{3(10+x)} = \frac{12x+70}{190}.$$

$$22. \frac{x}{x^2-1} = \frac{15-7x}{8(1-x)}.$$

$$25. x - \frac{14x-9}{8x-3} = \frac{x^2-3}{x+1}.$$

$$23. \frac{2x-1}{x-1} + \frac{1}{6} = \frac{2x-3}{x-2}.$$

$$26. 1 - \frac{x+5}{2x+1} = \frac{x-6}{x-2}.$$

$$24. \frac{x+2}{x-1} - \frac{4-x}{2x} = \frac{7}{3}.$$

$$27. \frac{x}{7-x} + \frac{7-x}{x} = 2\frac{2}{15}.$$

$$28. \frac{2x+3}{2(2x-1)} - \frac{7-x}{2(x+1)} = \frac{7-3x}{4-3x}.$$

$$29. \frac{12x^3-11x^2+10x-78}{8x^2-7x+6} = 1\frac{1}{2}x - \frac{1}{2}.$$

$$30. \frac{6}{x-1} - \frac{18}{x+5} = \frac{7}{x+1} - \frac{8}{x-5}.$$

227. *Literal quadratic equations* are solved as follows :

- (1) Solve the equation $ax^2 + bx = c$.

Multiply the equation by $4a$ and add the square of b ,

$$4a^2x^2 + () + b^2 = 4ac + b^2.$$

Extract the root,

$$2ax + b = \pm \sqrt{4ac + b^2}.$$

Reduce,

$$2ax = -b \pm \sqrt{4ac + b^2}.$$

$$\therefore x = \frac{-b \pm \sqrt{4ac + b^2}}{2a}.$$

- (2) Solve the equation $adx - acx^2 = bcx - bd$.

Transpose bcx and change the signs,

$$acx^2 + bcx - adx = bd.$$

Express the left member in *two terms*,

$$acx^2 + (bc - ad)x = bd.$$

Multiply by $4ac$,

$$4a^2c^2x^2 + 4ac(bc - ad)x = 4abcd.$$

Complete the square,

$$4a^2c^2x^2 + () + (bc - ad)^2 = b^2c^2 + 2abcd + a^2d^2.$$

Extract the root,

$$2acx + (bc - ad) = \pm (bc + ad).$$

Reduce,

$$2acx = -(bc - ad) \pm (bc + ad),$$

$$= 2ad \text{ or } -2bc.$$

$$\therefore x = \frac{d}{c} \text{ or } -\frac{b}{a}.$$

- (3) Solve the equation $px^2 - px + qx^2 + qx = \frac{pq}{p+q}$.

Express the left number in *two terms*,

$$(p+q)x^2 - (p-q)x = \frac{pq}{p+q}.$$

Multiply by four times the coefficient of x^2 ,

$$4(p+q)^2x^2 - 4(p^2 - q^2)x = 4pq.$$

Complete the square,

$$4(p+q)^2x^2 - () + (p-q)^2 = p^2 + 2pq + q^2.$$

Extract the root, $2(p+q)x - (p-q) = \pm (p+q)$.

Reduce,

$$2(p+q)x = (p-q) \pm (p+q),$$

$$= 2p \text{ or } -2q.$$

$$\therefore x = \frac{p}{p+q} \text{ or } -\frac{q}{p+q}.$$

NOTE. The left-hand member of the equation when simplified must be expressed in *two terms*, *simple or compound*, one term containing x^2 , and the other term containing x .

EXERCISE 85.

Solve:

1. $x^2 + 2ax = a^2$.

14. $x^2 + ax = a + x$.

2. $x^2 = 4ax + 7a^2$.

15. $x^2 + ax = bx + ab$.

3. $x^2 = \frac{7m^2}{4} - 3mx$.

16. $\frac{x}{a} + \frac{a}{x} = \frac{x}{b} + \frac{b}{x}$.

4. $x^2 - \frac{5nx}{2} - \frac{3n^2}{2} = 0$.

17. $\frac{1}{x} + \frac{1}{x+b} = \frac{1}{a} + \frac{1}{a+b}$.

5. $\frac{a^2}{(x+a)^2} = \frac{b^2}{(x-a)^2}$.

18. $\frac{a}{3} + \frac{5x}{4} - \frac{x^2}{3a} = 0$.

6. $cx = ax^2 + bx^2 - \frac{ac}{a+b}$.

19. $\frac{x+3}{x-3} = a + \frac{x-3}{x+3}$.

7. $\frac{a^2x^2}{b^2} + \frac{b^2}{c^2} = \frac{2ax}{c}$.

20. $mx^2 - 1 = \frac{x(m^2 - n^2)}{mn}$.

8. $(a^2 + 1)x = ax^2 + a$.

21. $(ax - b)(bx - a) = c^2$.

9. $\frac{a}{x-a} + \frac{b}{x-b} = \frac{2c}{x-c}$.

22. $\frac{ax+b}{bx+a} = \frac{mx+n}{nx+m}$.

10. $\frac{1}{a+b+x} = \frac{1}{a} + \frac{1}{b} + \frac{1}{x}$.

23. $\frac{m}{m+x} + \frac{m}{m-x} = c$.

11. $\frac{1}{a-x} - \frac{1}{a+x} = \frac{3+x^2}{a^2-x^2}$.

24. $\frac{(a-1)^2x^2 + 2(3a-1)x}{4a-1} = 1$.

12. $\frac{x^2 + 2ab(a^2 + b^2)}{a^2 + b^2} = 2x$.

25. $\frac{(a^2 - b^2)(x^2 + 1)}{a^2 + b^2} = 2x$.

13. $\frac{(2x-a)^2}{2x-a+2b} = b$.

26. $\frac{x^2 - 4mnx}{(m+n)^2} = (m-n)^2$.

27. $x^2 + \frac{a-b}{ab^2} = \frac{14a^2 - 5ab - 10b^2}{18a^2b^2} + \frac{(2a-3b)x}{2ab}$.

$$28. \quad abx^2 + \frac{b^2x}{c} = \frac{6a^2 + ab - 2b^2}{c^2} - \frac{3a^2x}{c}.$$

$$29. \quad \frac{x^2}{3m-2a} - \frac{m^2-4a^2}{4a-6m} = \frac{x}{2}.$$

$$30. \quad 6x + \frac{(a+b)^2}{x} = 5(a-b) + \frac{25ab}{6x}.$$

$$31. \quad \frac{8}{9}(x^2 + a^2 + ab) = \frac{1}{9}x(20a + 4b).$$

$$32. \quad x^2 - (b-a)c = ax - bx + cx.$$

$$33. \quad x^2 - 2mx = (n-p+m)(n-p-m).$$

$$34. \quad x^2 - (m+n)x = \frac{1}{4}(p+q+m+n)(p+q-m-n).$$

$$35. \quad mnx^2 - (m+n)(mn+1)x + (m+n)^2 = 0.$$

$$36. \quad \frac{2b-x-2a}{bx} + \frac{4b-7a}{ax-bx} = \frac{x-4a}{ab-b^2}.$$

$$37. \quad 2x^2(a^2-b^2) - (3a^2+b^2)(x-1) = (3b^2+a^2)(x+1).$$

$$38. \quad \frac{a-2b-x}{a^2-4b^2} - \frac{5b-x}{ax+2bx} + \frac{2a-x-19b}{2bx-ax} = 0.$$

$$39. \quad \frac{x+13a+3b}{5a-3b-x} - 1 = \frac{a-2b}{x+2b}.$$

$$40. \quad \frac{x+3b}{8a^2-12ab} - \frac{3b}{9b^2-4a^2} - \frac{a+3b}{(2a+3b)(x-3b)} = 0.$$

$$41. \quad nx^2 + px - px^2 - mx + m - n = 0.$$

$$42. \quad (a+b+c)x^2 - (2a+b+c)x + a = 0.$$

$$43. \quad (ax-b)(c-d) = (a-b)(cx-d)x.$$

$$44. \quad \frac{2x+1}{b} - \frac{1}{x} \left(\frac{1}{b} - \frac{2}{a} \right) = \frac{3x+1}{a}.$$

$$45. \quad \frac{1}{2x^2+x-1} + \frac{1}{2x^2-3x+1} = \frac{a}{2bx-b} - \frac{2bx+b}{ax^2-a}.$$

228. An affected quadratic may be reduced to the form $x^2 + px + q = 0$, in which p and q represent *any* numbers, positive or negative, integral or fractional.

Ex. Solve: $x^2 + px + q = 0$.

$$\begin{aligned} 4x^2 + (\quad) + p^2 &= p^2 - 4q, \\ 2x + p &= \pm \sqrt{p^2 - 4q}, \\ \therefore x &= -\frac{p}{2} \pm \frac{1}{2} \sqrt{p^2 - 4q}. \end{aligned}$$

By this formula, the values of x in an equation of the form $x^2 + px + q = 0$, may be written at once. Thus, take the equation

$$3x^2 - 5x + 2 = 0.$$

Divide by 3, $x^2 - \frac{5}{3}x + \frac{2}{3} = 0$.

Here,

$$p = -\frac{5}{3}, \text{ and } q = \frac{2}{3}.$$

$$\begin{aligned} \therefore x &= \frac{5}{6} \pm \frac{1}{2} \sqrt{\frac{25}{9} - \frac{8}{3}}, \\ &= \frac{5}{6} \pm \frac{1}{6}, \\ &= 1 \text{ or } \frac{2}{3}. \end{aligned}$$

229. A quadratic which has been reduced to its simplest form, and has all its terms written on one side, may often have that side resolved *by inspection* into factors.

In this case, the roots are seen at once without completing the square.

(1) Solve $x^2 + 7x - 60 = 0$.

Since $x^2 + 7x - 60 = (x + 12)(x - 5)$,
the equation $x^2 + 7x - 60 = 0$
may be written $(x + 12)(x - 5) = 0$.

It will be observed that if *either* of the factors $x + 12$ or $x - 5$ is 0, the *product of the two factors* is 0, and the equation is satisfied.

Hence, $x + 12 = 0$ and $x - 5 = 0$.
 $\therefore x = -12$, and $x = 5$.

(2) Solve $x^2 + 7x = 0$.

The equation $x^2 + 7x = 0$
 becomes $x(x + 7) = 0$,
 and is satisfied if $x = 0$, or if $x + 7 = 0$.

\therefore the roots are 0 and -7 .

It will be observed that this method is easily applied to an equation *all* the terms of which contain x .

(3) Solve $2x^2 - x^2 - 6x = 0$.

The equation $2x^2 - x^2 - 6x = 0$
 becomes $x(2x^2 - x - 6) = 0$,
 and is satisfied if $x = 0$, or if $2x^2 - x - 6 = 0$.

By solving $2x^2 - x - 6 = 0$ the two roots 2 and $-\frac{3}{2}$ are found.

\therefore the equation has *three* roots, 0, 2, $-\frac{3}{2}$.

(4) Solve $x^3 + x^2 - 4x - 4 = 0$.

The equation $x^3 + x^2 - 4x - 4 = 0$
 becomes $x^2(x + 1) - 4(x + 1) = 0$,
 $(x^2 - 4)(x + 1) = 0$.

\therefore the roots of the equation are -1 , 2, -2 .

(5) Solve $x^3 - 2x^2 - 11x + 12 = 0$.

Since $\frac{x^3 - 2x^2 - 11x + 12}{x - 1} = x^2 - x - 12$,
 the equation $x^3 - 2x^2 - 11x + 12 = 0$
 may be written $(x - 1)(x^2 - x - 12) = 0$.

The three roots are found to be 1, -3 , 4.

An equation which cannot be resolved into factors by inspection may sometimes be solved by *guessing* a root and reducing by division. In this case, if a denote the root, the given equation (all the terms of the equation being written on one side) may be divided by $x - a$.

EXERCISE 86.

Find the roots of :

1. $(x+1)(x-2)(x^2+x-2)=0$.
2. $(x^2-3x+2)(x^2-x-12)=0$.
3. $(x+1)(x-2)(x+3)=-6$.
4. $2x^3+4x^2-70x=0$.
5. $(x^2-x-6)(x^2-x-20)=0$.
6. $x(x+1)(x+2)=(a+2)(a+1)a$.
7. $x^3-x^2-x+1=0$.
8. $8x^3-1=0$.
9. $8x^3+1=0$.
10. $x^6-1=0$.
11. $x(x-a)(x^2-b^2)=0$.
12. $n(x^3+1)+x+1=0$.

230. If r and r' represent two values of x , then

$$x-r=0,$$

and

$$x-r'=0.$$

$$\therefore (x-r)(x-r')=0.$$

This is a quadratic equation, as may be seen by performing the indicated multiplication.

Now, r and r' are roots of this equation; for, if either r or r' be written for x , one of the factors, $x-r$, $x-r'$, is equal to 0, and the equation is satisfied. Also r and r' are the *only* roots, for no value of x , except r and r' , can make either of these factors equal to 0.

Since r and r' may represent the values of x in any quadratic equation, it follows that every quadratic equation has *two* roots, and *only two*.

Again, if r , r' , r'' , represent three values of x ,
then,

$$(x-r)(x-r')(x-r'')=0.$$

This is a *cubic* equation, as may be seen by performing the indicated multiplication. Hence, it may be inferred that a *cubic* equation has *three* roots, and *only three*; and so, for any equation, that the *number* of roots is equal to the *degree* of the equation.

It may also be inferred that if r be a root of an equation, $x-r$ will be a *factor* of the equation when the equation is written with all its terms on one side.

If r and r' represent the roots of the general quadratic equation,

$$x^2 + px + q = 0.$$

This equation may be written $(x - r)(x - r') = 0$,
 or,
$$x^2 - (r + r')x + rr' = 0.$$

A form which shows that
 the *sum* of the roots $= -p$,
 and the *product* of the roots $= q$.

231. It will be seen from § 230 that an equation may be formed if its roots are known.

If the roots of an equation be -1 and $\frac{1}{2}$,
 the equation will be
$$(x + 1)(x - \frac{1}{2}) = 0,$$

or,
$$x^2 + \frac{3x}{4} - \frac{1}{4} = 0,$$

or, by multiplying by 4,
$$4x^2 + 3x - 1 = 0.$$

If the roots of an equation be 0, 1, 5,
 the equation will be $(x - 0)(x - 1)(x - 5) = 0$;
 that is,
$$x(x - 1)(x - 5) = 0,$$

 or,
$$x^3 - 6x^2 + 5x = 0.$$

If x occur in *every* term, the equation will be satisfied by putting $x = 0$, and may be reduced to an equation of the next lower degree by dividing every term by x .

232. By considering the roots of $x^2 + px + q = 0$,

namely,
$$r = -\frac{p}{2} + \frac{1}{2}\sqrt{p^2 - 4q},$$

and
$$r' = -\frac{p}{2} - \frac{1}{2}\sqrt{p^2 - 4q},$$

it will be seen that the *character of the roots* of an equation may be determined without solving it:

I. As the two roots have the same expression, $\sqrt{p^2 - 4q}$,
both roots will be *real*, or *both* will be *imaginary*.

If both be real, *both* will be *rational* or *both surds*, according as $p^2 - 4q$ is or is not a perfect square.

II. When p^2 is greater than $4q$, the two roots will be *real*, for then the expression $p^2 - 4q$ is *positive*, and therefore $\sqrt{p^2 - 4q}$ can be found exactly or approximately.

Since also its value in one root is to be added to $-\frac{p}{2}$, and in the other to be subtracted from $-\frac{p}{2}$, the two roots will be *different in value*.

III. When p^2 is equal to $4q$, the roots will be *equal in value*.

IV. When p^2 is less than $4q$, the roots will be *imaginary*, for then the expression $p^2 - 4q$ will be negative, and therefore $\sqrt{p^2 - 4q}$ represents the *even* root of a *negative* number, and is imaginary.

V. If $q (= r \times r')$ be *positive*, the roots, if real, will have the *same sign*, but opposite to that of p (since $r + r' = -p$).

But if q be *negative*, the roots will have *opposite signs*.

233. Determine by inspection the character of the roots of:

(1) $x^2 - 5x + 6 = 0$.

In this equation p is -5 , and q is 6 .

$$\therefore \sqrt{p^2 - 4q} = \sqrt{25 - 24} = 1.$$

\therefore the roots will be *rational*, and *both positive*.

(2) $x^2 + 3x + 1 = 0$.

In this equation p is 3 , and q is 1 .

$$\therefore \sqrt{p^2 - 4q} = \sqrt{9 - 4} = \sqrt{5}.$$

\therefore the roots will be *surds*, and *both negative*.

(3) $x^2 + 3x + 4 = 0$.

In this equation p is 3 , and q is 4 .

$$\therefore \sqrt{p^2 - 4q} = \sqrt{9 - 16} = \sqrt{-7}.$$

\therefore the roots will be *imaginary*.

EXERCISE 87.

Form the equations whose roots are :

- | | | |
|----------------------------------|---|---|
| 1. 2, 1. | 5. $-5, -\frac{1}{2}$. | 9. $0, -\frac{1}{2}, \frac{3}{2}, -1$. |
| 2. 7, -3 . | 6. $-\frac{7}{6}, \frac{2}{3}$. | 10. $a-2b, 3a+2b$. |
| 3. $\frac{1}{2}, \frac{1}{3}$. | 7. $3, -3, \frac{2}{3}, -\frac{2}{3}$. | 11. $2a-b, b-3a$. |
| 4. $\frac{2}{3}, -\frac{2}{3}$. | 8. $0, 1, 2, 3$. | 12. $a(a+1), 1-a$. |

Determine by inspection the character of the roots of :

- | | |
|---------------------------|---------------------------|
| 13. $x^2 - 7x + 12 = 0$. | 17. $x^2 + 4x + 1 = 0$. |
| 14. $x^2 - 7x - 30 = 0$. | 18. $x^2 - 2x + 9 = 0$. |
| 15. $x^2 + 4x - 5 = 0$. | 19. $3x^2 - 4x - 4 = 0$. |
| 16. $5x^2 + 8 = 0$. | 20. $x^2 + 4x + 4 = 0$. |

234. It is often useful to determine the maximum or minimum value of a given *quadratic expression*.

- (1) Find the maximum or minimum value of $1+x-x^2$.

$$\begin{array}{ll}
 \text{Let} & 1+x-x^2=m; \\
 \text{then,} & x^2-x=1-m, \\
 \text{and} & 4x^2-(\quad)+1=5-4m, \\
 & 2x-1=\pm\sqrt{5-4m}. \\
 & \therefore x=\frac{1}{2}\pm\frac{1}{2}\sqrt{5-4m}.
 \end{array}$$

Now, for all possible values of x , $5-4m$ cannot be negative; that is, m cannot be *greater* than $\frac{5}{4}$; and for this value x is $\frac{1}{2}$. Therefore, $\frac{5}{4}$ is the *maximum* value of the given expression.

- (2) Find the maximum or minimum value of x^2+3x+4 .

$$\begin{array}{ll}
 \text{Let} & x^2+3x+4=m; \\
 \text{then,} & x^2+3x=m-4, \\
 \text{and} & 4x^2+(\quad)+9=4m-7, \\
 & 2x+3=\pm\sqrt{4m-7}. \\
 & x=-\frac{3}{2}\pm\frac{1}{2}\sqrt{4m-7}.
 \end{array}$$

For all possible values of x , $4m-7$ cannot be negative; that is, m cannot be *less* than $\frac{7}{4}$; and for this value $x=-\frac{3}{2}$. Therefore, $\frac{7}{4}$ is the *minimum* value of the given expression.

EXERCISE 88.

Find the maximum or minimum value (and determine which) of:

- | | | |
|----------------------------|--------------------------|-----------------------------------|
| 1. $4 + 6x - x^2$. | 4. $(a - x)(x - b)$. | 7. $x^2 - 2x + 9$. |
| 2. $\frac{(x + a)^2}{x}$. | 5. $\frac{x}{1 + x^2}$. | 8. $\frac{x^2}{(x + a)(x - b)}$. |
| 3. $\frac{x^2 + 1}{x}$. | 6. $x^2 + 8x + 20$. | 9. $\frac{x}{a + x^2}$. |

10. Divide a line 20 in. long into two parts so that the sum of the squares on these two parts may be the least possible.
11. Divide a line 20 in. long into two parts so that the rectangle contained by the parts may be the greatest possible.
12. Find the fraction which has the greatest excess over its square.

235. Two other cases of the solution of equations *by completing the square* should be noticed.

I. When *any* two powers of x are involved, *one of which is the square of the other*.

II. When the *addition of a number* to an equation of the fourth degree will *make both sides complete squares*.

- (1) Solve $8x^6 + 63x^3 = 8$.

In this equation the exponent 6 is the double of 3, hence x^3 is the square of $x^{\frac{3}{2}}$.

$$\begin{aligned} 8x^6 + 63x^3 &= 8, \\ 256x^6 + () + (63)^2 &= 4225, \\ 16x^3 + 63 &= \pm 65, \\ 16x^3 &= 2, \text{ or } -128, \\ x^3 &= \frac{1}{8}, \text{ or } -8. \end{aligned}$$

By taking cube root, $x = \frac{1}{2}, \text{ or } -2.$

The other roots of the equation are found by finding the remaining roots of the equations, $x^3 = \frac{1}{8}$, and $x^3 = -8$.

Since, $x^3 = \frac{1}{8}$, $\therefore 8x^3 - 1 = 0$	Since, $x^3 = -8$, $\therefore x^3 + 8 = 0$
Now, by § 230,	Now, by § 230,
$8x^3 - 1 = (2x - 1)(4x^2 + 2x + 1)$	$x^3 + 8 = (x + 2)(x^2 - 2x + 4)$
$\therefore (2x - 1)(4x^2 + 2x + 1) = 0$	$\therefore (x + 2)(x^2 - 2x + 4) = 0$
and is satisfied if $4x^2 + 2x + 1 = 0$	and is satisfied if $x^2 - 2x + 4 = 0$
as well as if $2x - 1 = 0$.	as well as if $x + 2 = 0$
The solution of $4x^2 + 2x + 1 = 0$	The solution of $x^2 - 2x + 4 = 0$
gives $x = \frac{1}{2}(-1 \pm \sqrt{-3})$.	gives $x = 1 \pm \sqrt{-3}$.

\therefore the roots are $\frac{1}{2}, -2, 1 \pm \sqrt{-3}, \frac{1}{2}(-1 \pm \sqrt{-3})$.

(2) Solve $x^4 - 10x^3 + 35x^2 - 50x + 24 = 0$.

Take the square root of the left side.

$$\begin{array}{r}
 x^4 - 10x^3 + 35x^2 - 50x + 24 \overline{) x^2 - 5x + 5} \\
 \underline{2x^2 - 5x} \quad \begin{array}{l} x^4 \\ 2x^2 - 5x \end{array} \quad \begin{array}{l} -10x^3 + 35x^2 \\ -10x^3 + 25x^2 \end{array} \\
 \underline{2x^2 - 10x + 5} \quad \begin{array}{l} 10x^3 - 50x + 24 \\ 10x^3 - 50x + 25 \end{array} \\
 \quad \quad \quad \underline{ - 1}
 \end{array}$$

It is now seen that if 1 were added, the square would be complete and the equation would be

$$x^4 - 10x^3 + 35x^2 - 50x + 25 = 1.$$

Extract the square root, and the result is,

$$\begin{array}{l}
 x^2 - 5x + 5 = \pm 1. \\
 \text{That is,} \quad \quad \quad x^2 - 5x = -4, \text{ or } -6. \\
 4x^2 - () + 25 = 9, \text{ or } 1, \\
 \quad \quad \quad 2x - 5 = \pm 3, \text{ or } \pm 1, \\
 \quad \quad \quad 2x = 8, 2, 6, \text{ or } 4. \\
 \therefore x = 4, 1, 3, \text{ or } 2.
 \end{array}$$

EXERCISE 89.

Find the roots of:

1. $x^6 + 7x^3 = 8$.
2. $x^4 - 5x^2 + 4 = 0$.
3. $37x^2 - 9 = 4x^4$.
4. $16x^8 = 17x^4 - 1$.
5. $32x^{10} - 33x^5 + 1 = 0$.
6. $(x^2 - 2)^2 = \frac{1}{4}(x^2 + 12)$.
7. $x^{4n} - \frac{5x^{2n}}{3} - \frac{25}{12} = 0$.
8. $(x^2 - 9)^2 = 3 + 11(x^2 - 2)$.
9. $x^6 + 14x^3 + 24 = 0$.
10. $19x^4 + 216x^7 = x$.
11. $x^8 + 22x^4 + 21 = 0$.
12. $x^{2n} + 3x^n - 4 = 0$.
13. $4x^4 - 20x^3 + 23x^2 + 5x = 6$.
14. $\frac{1}{x^{2n}} + \frac{3}{x^n} - 20 = 0$.
15. $x^4 - 4x^3 - 10x^2 + 28x - 15 = 0$.
16. $x^4 - 2x^3 - 13x^2 + 14x = -24$.
17. $108x^4 = 20x(9x^2 - 1) - 51x^2 + 7$.
18. $(x^2 - 1)(x^2 - 2) + (x^2 - 3)(x^2 - 4) = x^4 + 5$.

PROBLEMS INVOLVING QUADRATICS.

236. Problems which involve quadratic equations have apparently *two* solutions, as a quadratic has *two* roots. Sometimes both will be solutions: but generally one only will be a solution, and the other be inconsistent with the conditions of the problem. No difficulty will be found in selecting the result which belongs to the problem, and sometimes a change may be made in the statement of a problem so as to form a new problem corresponding to the solution which was inapplicable to the original problem.

- (1) The sum of the squares of two consecutive numbers is 481. Find the numbers.

Let $x =$ one number,
 and $x + 1 =$ the other.
 Then $x^2 + (x + 1)^2 = 481$,
 or $2x^2 + 2x + 1 = 481$.

The solution of which gives, $x = 15$, or -16 .

The positive root 15 gives for the numbers, 15 and 16.

The negative root -16 is inapplicable to the problem, as *consecutive numbers* are understood to be integers which follow one another in the common scale, 1, 2, 3, 4.....

- (2) What is the price of eggs per dozen when 2 more in a shilling's worth lowers the price 1 penny per dozen?

Let $x =$ the number of eggs for a shilling.
 Then, $\frac{1}{x} =$ the cost of 1 egg in shillings,
 and $\frac{12}{x} =$ the cost of 1 dozen in shillings.
 But, if $x + 2 =$ the number of eggs for a shilling,
 $\frac{12}{x + 2} =$ the cost of 1 dozen in shillings.
 $\therefore \frac{12}{x} - \frac{12}{x + 2} = \frac{1}{12}$ (1 penny being $\frac{1}{12}$ of a shilling).

The solution of which gives $x = 16$, or -18 .

And, if 16 eggs cost a shilling, 1 dozen will cost $1\frac{2}{3}$ of a shilling, or 9 pence.

Therefore, the price of the eggs is 9 pence per dozen.

If the problem is changed so as to read: What is the price of eggs per dozen when two *less* in a shilling's worth *raises* the price 1 penny per dozen? the algebraic statement will be

$$\frac{12}{x - 2} - \frac{12}{x} = \frac{1}{12}.$$

The solution of which gives $x = 18$, or -16 .

Hence, the number 18, which had a negative sign and was inapplicable in the original problem, is here the true result.

EXERCISE 90.

1. The sum of the squares of three consecutive numbers is 365. Find the numbers.
2. Three times the product of two consecutive numbers exceeds four times their sum by 8. Find the numbers.
3. The product of three consecutive numbers is equal to three times the middle number. Find the numbers.
4. A boy bought a number of apples for 16 cents. Had he bought 4 more for the same money he would have paid $\frac{1}{3}$ of a cent less for each apple. How many did he buy?
5. For building 108 rods of stone-wall, 6 days less would have been required if 3 rods more a day had been built. How many rods a day were built?
6. A merchant bought some pieces of silk for \$900. Had he bought 3 pieces more for the same money he would have paid \$15 less for each piece. How many did he buy?
7. A merchant bought some pieces of cloth for \$168.75. He sold the cloth for \$12 a piece and gained as much as 1 piece cost him. How much did he pay for each piece?
8. Find the price of eggs per score when 10 more in 62 $\frac{1}{2}$ cents' worth lowers the price 31 $\frac{1}{4}$ cents per hundred.
9. The area of a square may be doubled by increasing its length by 6 inches and its breadth by 4 inches. Determine its side.
10. The length of a rectangular field exceeds the breadth by 1 yard, and the area is 3 acres. Determine its dimensions.

11. There are three lines of which two are each $\frac{1}{2}$ of the third, and the sum of the squares described on them is equal to a square yard. Determine the lengths of the lines in inches.
12. A grass plot 9 yards long and 6 yards broad has a path round it. The area of the path is equal to that of the plot. Determine the width of the path.
13. Find the radius of a circle the area of which would be doubled by increasing its radius by 1 inch.
14. Divide a line 20 inches long into two parts so that the rectangle contained by the whole and one part may be equal to the square on the other part.
15. A can do some work in 9 hours less time than B can do it, and together they can do it in 20 hours. How long will it take each alone to do it?
16. A vessel which has two pipes can be filled in two hours less time by one than by the other, and by both together in 2 hours 55 minutes. How long will it take each pipe alone to fill the vessel?
17. A vessel which has two pipes can be filled in 2 hours less time by one than by the other, and by both together in 1 hour 52 minutes 30 seconds. How long will it take each pipe alone to fill the vessel?
18. An iron bar weighs 36 pounds. If it had been 1 foot longer each foot would have weighed $\frac{1}{2}$ a pound less. Find the length and the weight per foot.
19. A number is expressed by two digits, the second of which is the square of the other, and when 54 is added its digits are interchanged. Find the number.
20. Divide 35 into two parts so that the sum of the two fractions formed by dividing each part by the other may be $2\frac{1}{2}$.

21. A boat's crew row $3\frac{1}{2}$ miles down a river and back again in 1 hour 40 minutes. If the current of the river is 2 miles per hour, determine their rate of rowing in still water.
22. A detachment from an army was marching in regular column with 5 men more in depth than in front. On approaching the enemy the front was increased by 845 men, and the whole was thus drawn up in 5 lines. Find the number of men.
23. A jockey sold a horse for \$144, and gained as much per cent as the horse cost. What did the horse cost?
24. A merchant expended a certain sum of money in goods, which he sold again for \$24, and lost as much per cent as the goods cost him. How much did he pay for the goods?
25. A broker bought a number of bank shares (\$100 each), when they were at a certain per cent *discount*, for \$7500; and afterwards when they were at the same per cent *premium*, sold all but 60 for \$5000. How many shares did he buy, and at what price?
26. The thickness of a rectangular solid is $\frac{2}{3}$ of its width, and its length is equal to the sum of its width and thickness; also, the number of cubic yards in its volume added to the number of linear yards in its edges is $\frac{4}{3}$ of the number of square yards in its surface. Determine its dimensions.
27. If a carriage-wheel $16\frac{1}{2}$ feet round took 1 second more to revolve, the rate of the carriage per hour would be $1\frac{1}{4}$ miles less. At what rate is the carriage travelling?

CHAPTER XV.

SIMULTANEOUS QUADRATIC EQUATIONS.

237. QUADRATIC equations involving *two* unknown quantities require different methods for their solution, according to the *form* of the equations.

238. CASE I. When from one of the equations the value of one of the unknown quantities can be found in terms of the other, and this value *substituted* in the other equation.

$$\begin{array}{rcl} \text{Ex. Solve:} & 3x^2 - 2xy = 5 & (1) \\ & x - y = 2 & (2) \end{array}$$

$$\begin{array}{ll} \text{Transpose } x \text{ in (2),} & y = x - 2. \\ \text{Substitute in (1),} & 3x^2 - 2x(x - 2) = 5. \\ \text{The solution of which gives} & x = 1 \text{ or } -5. \\ & \therefore y = -1 \text{ or } -7. \end{array}$$

Special methods often give more elegant solutions of examples than the *general* method by *substitution*.

I. When equations have the form, $x \pm y = a$, and $xy = b$; $x^2 \pm y^2 = a$, and $xy = b$; or, $x \pm y = a$, and $x^2 + y^2 = b$.

$$\begin{array}{rcl} (1) \text{ Solve:} & x + y = 40 & (1) \\ & xy = 300 & (2) \end{array}$$

$$\text{Square (1),} \quad x^2 + 2xy + y^2 = 1600. \quad (3)$$

$$\text{Multiply (2) by 4,} \quad \frac{4xy}{\quad} = 1200. \quad (4)$$

$$\text{Subtract (4) from (3),} \quad x^2 - 2xy + y^2 = 400.$$

$$\text{Extract root of each side,} \quad x - y = \pm 20. \quad (5)$$

$$\text{Add (1) and (5),} \quad 2x = 60 \text{ or } 20.$$

$$\therefore x = 30 \text{ or } 10.$$

$$\text{Subtract (5) from (1),} \quad 2y = 20 \text{ or } 60.$$

$$\therefore y = 10 \text{ or } 30.$$

$$(2) \text{ Solve: } \left. \begin{array}{l} x - y = 4 \\ x^2 + y^2 = 40 \end{array} \right\} \quad \begin{array}{l} (1) \\ (2) \end{array}$$

$$\text{Square (1),} \quad x^2 - 2xy + y^2 = 16. \quad (3)$$

$$\text{Subtract (2) from (3),} \quad -2xy = -24. \quad (4)$$

$$\text{Subtract (4) from (2),} \quad x^2 + 2xy + y^2 = 64. \quad (5)$$

$$\text{Extract the root,} \quad x + y = \pm 8. \quad (5)$$

$$\text{By combining (5) and (1),} \quad \begin{array}{l} x = 6 \text{ or } -2. \\ y = 2 \text{ or } -6. \end{array}$$

$$(3) \text{ Solve: } \left. \begin{array}{l} \frac{1}{x} + \frac{1}{y} = \frac{9}{20} \\ \frac{1}{x^2} + \frac{1}{y^2} = \frac{41}{400} \end{array} \right\} \quad \begin{array}{l} (1) \\ (2) \end{array}$$

$$\text{Square (1),} \quad \frac{1}{x^2} + \frac{2}{xy} + \frac{1}{y^2} = \frac{81}{400}. \quad (3)$$

$$\text{Subtract (2) from (3)} \quad \frac{2}{xy} = \frac{40}{400}. \quad (4)$$

$$\text{Subtract (4) from (2),} \quad \frac{1}{x^2} - \frac{2}{xy} + \frac{1}{y^2} = \frac{1}{400}.$$

$$\text{Extract the root,} \quad \frac{1}{x} - \frac{1}{y} = \pm \frac{1}{20}. \quad (5)$$

$$\text{By combining (1) and (5),} \quad \begin{array}{l} x = 4 \text{ or } 5. \\ y = 5 \text{ or } 4. \end{array}$$

II. When one equation may be simplified by dividing it by the other.

$$(4) \text{ Solve: } \left. \begin{array}{l} x^3 + y^3 = 91 \\ x + y = 7 \end{array} \right\} \quad \begin{array}{l} (1) \\ (2) \end{array}$$

$$\text{Divide (1) by (2),} \quad x^2 - xy + y^2 = 13. \quad (3)$$

$$\text{Square (2),} \quad x^2 + 2xy + y^2 = 49. \quad (4)$$

$$\text{Subtract (3) from (4),} \quad 3xy = 36.$$

$$\text{Divide by } -3, \quad -xy = -12. \quad (5)$$

$$\text{Add (5) and (3).} \quad x^2 - 2xy + y^2 = 1. \quad (6)$$

$$\text{Extract the root,} \quad x - y = \pm 1. \quad (6)$$

$$\text{By combining (6) and (2),} \quad \begin{array}{l} x = 4 \text{ or } 3. \\ y = 3 \text{ or } 4. \end{array}$$

EXERCISE 91.

Solve :

- | | |
|---|---|
| 1. $\left. \begin{array}{l} x+y=13 \\ xy=36 \end{array} \right\}$ | 11. $\left. \begin{array}{l} x+y=49 \\ x^2+y^2=1681 \end{array} \right\}$ |
| 2. $\left. \begin{array}{l} x+y=29 \\ xy=100 \end{array} \right\}$ | 12. $\left. \begin{array}{l} x^3+y^3=341 \\ x+y=11 \end{array} \right\}$ |
| 3. $\left. \begin{array}{l} x-y=19 \\ xy=66 \end{array} \right\}$ | 13. $\left. \begin{array}{l} x^3+y^3=1008 \\ x+y=12 \end{array} \right\}$ |
| 4. $\left. \begin{array}{l} x-y=45 \\ xy=250 \end{array} \right\}$ | 14. $\left. \begin{array}{l} x^3-y^3=98 \\ x-y=2 \end{array} \right\}$ |
| 5. $\left. \begin{array}{l} x-y=10 \\ x^2+y^2=178 \end{array} \right\}$ | 15. $\left. \begin{array}{l} x^3-y^3=279 \\ x-y=3 \end{array} \right\}$ |
| 6. $\left. \begin{array}{l} x-y=14 \\ x^2+y^2=436 \end{array} \right\}$ | 16. $\left. \begin{array}{l} x-3y=1 \\ xy+y^2=5 \end{array} \right\}$ |
| 7. $\left. \begin{array}{l} x+y=12 \\ x^2+y^2=104 \end{array} \right\}$ | 17. $\left. \begin{array}{l} 4y=5x+1 \\ 2xy=33-x^2 \end{array} \right\}$ |
| 8. $\left. \begin{array}{l} \frac{1}{x}+\frac{1}{y}=\frac{3}{4} \\ \frac{1}{x^2}+\frac{1}{y^2}=\frac{5}{16} \end{array} \right\}$ | 18. $\left. \begin{array}{l} \frac{1}{x}-\frac{1}{y}=3 \\ \frac{1}{x^2}-\frac{1}{y^2}=21 \end{array} \right\}$ |
| 9. $\left. \begin{array}{l} \frac{1}{x}+\frac{1}{y}=5 \\ \frac{1}{x^2}+\frac{1}{y^2}=13 \end{array} \right\}$ | 19. $\left. \begin{array}{l} \frac{1}{x}-\frac{1}{y}=2\frac{1}{2} \\ \frac{1}{x^2}-\frac{1}{y^2}=8\frac{3}{4} \end{array} \right\}$ |
| 10. $\left. \begin{array}{l} 7x^2-8xy=159 \\ 5x+2y=7 \end{array} \right\}$ | 20. $\left. \begin{array}{l} x^2-2xy-y^2=1 \\ x+y=2 \end{array} \right\}$ |

239. CASE II. When each of the two equations is *homogeneous* and of the *second degree*.

$$\text{Ex. Solve: } \left. \begin{aligned} 2y^2 - 4xy + 3x^2 &= 17 \\ y^2 - x^2 &= 16 \end{aligned} \right\} \quad \begin{array}{l} (1) \\ (2) \end{array}$$

Let $y = vx$, and substitute vx for y in both equations.

$$\text{From (1),} \quad 2v^2x^2 - 4vx^2 + 3x^2 = 17.$$

$$\therefore x^2 = \frac{17}{2v^2 - 4v + 3}.$$

$$\text{From (2),} \quad v^2x^2 - x^2 = 16.$$

$$\therefore x^2 = \frac{16}{v^2 - 1}.$$

$$\text{Equate the values of } x^2, \quad \frac{17}{2v^2 - 4v + 3} = \frac{16}{v^2 - 1},$$

$$32v^2 - 64v + 48 = 17v^2 - 17,$$

$$15v^2 - 64v = -65.$$

$$\text{The solution gives,} \quad v = \frac{5}{3} \text{ or } \frac{13}{5}.$$

$$\text{Substitute the value of } v \text{ in} \quad x^2 = \frac{16}{v^2 - 1},$$

$$\text{then,} \quad x^2 = 9 \text{ or } \frac{25}{9}.$$

$$\therefore x = \pm 3 \text{ or } \pm \frac{5}{3},$$

$$\text{and} \quad y = vx = \pm 5 \text{ or } \pm \frac{13}{3}.$$

EXERCISE 92.

Solve:

$$1. \left. \begin{aligned} x^2 + xy + 2y^2 &= 74 \\ 2x^2 + 2xy + y^2 &= 73 \end{aligned} \right\}$$

$$2. \left. \begin{aligned} x^2 + xy + 4y^2 &= 6 \\ 3x^2 + 8y^2 &= 14 \end{aligned} \right\}$$

$$3. \left. \begin{aligned} x^2 - xy + y^2 &= 21 \\ y^2 - 2xy &= -15 \end{aligned} \right\}$$

$$4. \left. \begin{aligned} x^2 - 4y^2 - 9 &= 0 \\ xy + 2y^2 - 3 &= 0 \end{aligned} \right\}$$

$$5. \left. \begin{aligned} x^2 - xy - 35 &= 0 \\ xy + y^2 - 18 &= 0 \end{aligned} \right\}$$

$$6. \left. \begin{aligned} x^2 + xy + 2y^2 &= 44 \\ 2x^2 - xy + y^2 &= 16 \end{aligned} \right\}$$

$$\begin{array}{ll} 7. \left. \begin{array}{l} x^2 + xy - 15 = 0 \\ xy - y^2 - 2 = 0 \end{array} \right\} & 9. \left. \begin{array}{l} 2x^2 + 3xy + y^2 = 70 \\ 6x^2 + xy - y^2 = 50 \end{array} \right\} \end{array}$$

$$\begin{array}{ll} 8. \left. \begin{array}{l} x^2 - xy + y^2 = 7 \\ 3x^2 + 13xy + 8y^2 = 162 \end{array} \right\} & 10. \left. \begin{array}{l} x^2 - xy - y^2 = 5 \\ 2x^2 + 3xy + y^2 = 28 \end{array} \right\} \end{array}$$

240. CASE III. When the two equations are *symmetrical* with respect to x and y ; that is, when they have x and y similarly involved in them.

Thus, the expressions $2x^3 + 3x^2y^2 + 2y^3$, $2xy - 3x - 3y + 1$, $x^4 - 3x^2y - 3xy^2 + y^4$ are symmetrical expressions.

$$\begin{array}{ll} (1) \text{ Solve: } & \left. \begin{array}{l} x^3 + y^3 = 18xy \\ x + y = 12 \end{array} \right\} \end{array} \quad \begin{array}{l} (1) \\ (2) \end{array}$$

Put $u + v$ for x , and $u - v$ for y , in (1) and (2).

$$\begin{array}{ll} (1) \text{ becomes} & (u + v)^3 + (u - v)^3 = 18(u + v)(u - v), \\ \text{or} & u^3 + 3uv^2 = 9(u^2 - v^2). \end{array} \quad (3)$$

$$\begin{array}{ll} (2) \text{ becomes} & (u + v) + (u - v) = 12, \\ \text{or} & 2u = 12. \\ & \therefore u = 6. \end{array}$$

Substitute 6 for u in (3).

$$\begin{array}{ll} (3) \text{ becomes} & 216 + 18v^2 = 9(36 - v^2), \\ \text{whence,} & v^2 = 4. \end{array}$$

$$\begin{array}{ll} & \therefore v = \pm 2, \\ & \therefore x = u + v = 6 \pm 2 = 8 \text{ or } 4, \\ \text{and} & y = u - v = 6 \mp 2 = 4 \text{ or } 8. \end{array}$$

$$\begin{array}{ll} (2) \text{ Solve: } & \left. \begin{array}{l} x + y = 8 \\ x^4 + y^4 = 706 \end{array} \right\} \end{array} \quad \begin{array}{l} (1) \\ (2) \end{array}$$

Put $u + v$ for x , and $u - v$ for y in (1) and (2).

$$\begin{array}{ll} (1) \text{ becomes} & (u + v) + (u - v) = 8. \\ & \therefore u = 4. \end{array}$$

$$(2) \text{ becomes} \quad u^4 + 6u^2v^2 + v^4 = 353. \quad (3)$$

Substitute 4 for u in (3),

$$\begin{array}{ll} & 256 + 96v^2 + v^4 = 353, \\ \text{or,} & v^4 + 96v^2 = 97. \end{array} \quad (4)$$

The solution of (4) gives $v = \pm 1$ or $\pm \sqrt{-97}$.

Taking the possible values of v , $x = 5$ or 3 , and $y = 3$ or 5 .

EXERCISE 93.

Solve :

- | | |
|--|--|
| 1. $\left. \begin{aligned} 4xy &= 96 - x^2y^2 \\ x + y &= 6 \end{aligned} \right\}$ | 4. $\left. \begin{aligned} 4(x + y) &= 3xy \\ x + y + x^2 + y^2 &= 26 \end{aligned} \right\}$ |
| 2. $\left. \begin{aligned} x^2 + y^2 &= 18 - x - y \\ xy &= 6 \end{aligned} \right\}$ | 5. $\left. \begin{aligned} 4x^2 + xy + 4y^2 &= 58 \\ 5x^2 + 5y^2 &= 65 \end{aligned} \right\}$ |
| 3. $\left. \begin{aligned} 2(x^2 + y^2) &= 5xy \\ 4(x - y) &= xy \end{aligned} \right\}$ | 6. $\left. \begin{aligned} xy(x + y) &= 30 \\ x^3 + y^3 &= 35 \end{aligned} \right\}$ |

241. The preceding cases are *general methods* for the solution of equations which belong to the kinds referred to; often, however, in the solution of these and other kinds of simultaneous equations involving quadratics, a little ingenuity will suggest some step by which the roots may easily be found.

EXERCISE 94.

Solve :

- | | |
|--|---|
| 1. $\left. \begin{aligned} x - y &= 7 \\ x^2 + xy + y^2 &= 13 \end{aligned} \right\}$ | 8. $\left. \begin{aligned} x - y &= 1 \\ x^2 + y^2 &= 8\frac{1}{2} \end{aligned} \right\}$ |
| 2. $\left. \begin{aligned} x^2 + xy &= 35 \\ xy - y^2 &= 6 \end{aligned} \right\}$ | 9. $\left. \begin{aligned} x^2 + 4xy &= 3 \\ 4xy + y^2 &= 2\frac{1}{4} \end{aligned} \right\}$ |
| 3. $\left. \begin{aligned} xy - 12 &= 0 \\ x - 2y &= 5 \end{aligned} \right\}$ | 10. $\left. \begin{aligned} x^2 - xy + y^2 &= 48 \\ x - y - 8 &= 0 \end{aligned} \right\}$ |
| 4. $\left. \begin{aligned} xy - 7 &= 0 \\ x^2 + y^2 &= 50 \end{aligned} \right\}$ | 11. $\left. \begin{aligned} x^2 + 3xy + y^2 &= 1 \\ 3x^2 + xy + 3y^2 &= 13 \end{aligned} \right\}$ |
| 5. $\left. \begin{aligned} 2x - 5y &= 9 \\ x^2 - xy + y^2 &= 7 \end{aligned} \right\}$ | 12. $\left. \begin{aligned} x^2 - 2xy + 3y^2 &= 1\frac{2}{3} \\ x^2 + xy - y^2 &= \frac{1}{3} \end{aligned} \right\}$ |
| 6. $\left. \begin{aligned} x - y &= 9 \\ xy + 8 &= 0 \end{aligned} \right\}$ | 13. $\left. \begin{aligned} x + y &= a \\ 4xy - a^2 &= -4b^2 \end{aligned} \right\}$ |
| 7. $\left. \begin{aligned} 5x - 7y &= 0 \\ 5x^2 - \frac{13xy}{4} &= 4 - 7y^2 \end{aligned} \right\}$ | 14. $\left. \begin{aligned} x - y &= 1 \\ \frac{x}{y} + \frac{y}{x} &= 2\frac{1}{2} \end{aligned} \right\}$ |

15. $\left. \begin{aligned} x^2 + 9xy &= 340 \\ 7xy - y^2 &= 171 \end{aligned} \right\}$
16. $\left. \begin{aligned} x + y &= 6 \\ x^3 + y^3 &= 72 \end{aligned} \right\}$
17. $\left. \begin{aligned} 3xy + 2x + y &= 485 \\ 3x - 2y &= 0 \end{aligned} \right\}$
18. $\left. \begin{aligned} x - y &= 1 \\ x^3 - y^3 &= 19 \end{aligned} \right\}$
19. $\left. \begin{aligned} x^3 + y^3 &= 2728 \\ x^2 - xy + y^2 &= 124 \end{aligned} \right\}$
20. $\left. \begin{aligned} x + y &= a \\ x^2 + y^2 &= b^2 \end{aligned} \right\}$
21. $\left. \begin{aligned} x^2 - y^2 &= 0 \\ 3x^2 - 4xy + 5y^2 &= 9 \end{aligned} \right\}$
22. $\left. \begin{aligned} \frac{x+y}{x-y} + \frac{x-y}{x+y} &= \frac{10}{3} \\ x^2 + y^2 &= 45 \end{aligned} \right\}$
23. $\left. \begin{aligned} \frac{1}{x} + \frac{1}{y} &= 5 \\ \frac{1}{x+1} + \frac{1}{y+1} &= \frac{17}{12} \end{aligned} \right\}$
24. $\left. \begin{aligned} x^3 - xy + y^2 &= 7 \\ x^4 + x^2y^2 + y^4 &= 133 \end{aligned} \right\}$
25. $\left. \begin{aligned} x + y &= 4 \\ x^4 + y^4 &= 82 \end{aligned} \right\}$
26. $\left. \begin{aligned} x^3 - y^3 &= a^3 \\ x - y &= a \end{aligned} \right\}$
27. $\left. \begin{aligned} x^2 - xy &= a^2 + b^2 \\ xy - y^2 &= 2ab \end{aligned} \right\}$
28. $\left. \begin{aligned} x^2 - y^2 &= 4ab \\ xy &= a^2 - b^2 \end{aligned} \right\}$
29. $\left. \begin{aligned} xy &= 0 \\ x^2 + y^2 &= 16 \end{aligned} \right\}$
30. $\left. \begin{aligned} x^2 + xy + y^2 &= 37 \\ x^4 + x^2y^2 + y^4 &= 481 \end{aligned} \right\}$
31. $\left. \begin{aligned} x^2 &= ax + by \\ y^2 &= ay + bx \end{aligned} \right\}$
32. $\left. \begin{aligned} x - y - 2 &= 0 \\ 15(x^2 - y^2) &= 16xy \end{aligned} \right\}$
33. $\left. \begin{aligned} \frac{x+y}{x-y} + \frac{x-y}{x+y} &= \frac{89}{40} \\ 6x &= 20y + 9 \end{aligned} \right\}$
34. $\left. \begin{aligned} \frac{x}{a} + \frac{y}{b} &= 1 \\ \frac{a}{x} + \frac{b}{y} &= 4 \end{aligned} \right\}$
35. $\left. \begin{aligned} x^2 + y^2 &= 7 + xy \\ x^3 + y^3 &= 6xy - 1 \end{aligned} \right\}$
36. $\left. \begin{aligned} x^5 - y^5 &= 3093 \\ x - y &= 3 \end{aligned} \right\}$
37. $\left. \begin{aligned} \frac{3}{8}(x-1) - \frac{3}{8}(x+1)(y-1) &= -11 \\ \frac{1}{8}(y+2) &= \frac{1}{4}(x+2) \end{aligned} \right\}$
38. $\left. \begin{aligned} 10x^2 + 15xy &= 3ab - 2a^2 \\ 10y^2 + 15xy &= 3ab - 2b^2 \end{aligned} \right\}$

EXERCISE 95.

1. If the length and breadth of a rectangle were each increased by 1, the area would be 48; if they were each diminished by 1, the area would be 24. Find the length and breadth.
2. The sum of the squares of the two digits of a number is 25, and the product of the digits is 12. Find the number.
3. The sum, the product, and the difference of the squares of two numbers are all equal. Find the numbers.

NOTE. Represent the numbers by $x + y$ and $x - y$, respectively.

4. The difference of two numbers is $\frac{3}{8}$ of the greater, and the sum of their squares is 356. What are the numbers?
5. The numerator and denominator of one fraction are each greater by 1 than those of another, and the sum of the two fractions is $1\frac{5}{12}$; if the numerators were interchanged the sum of the fractions would be $1\frac{1}{2}$. Find the fractions.
6. A man starts from the foot of a mountain to walk to its summit. His rate of walking during the second half of the distance is $\frac{1}{2}$ mile per hour less than his rate during the first half, and he reaches the summit in $5\frac{1}{2}$ hours. He descends in $3\frac{3}{4}$ hours, by walking 1 mile more per hour than during the first half of the ascent. Find the distance to the top and the rates of walking.

NOTE. Let $2x =$ the distance, and y miles per hour = the rate at first.

$$\text{Then } \frac{x}{y} + \frac{x}{y - \frac{1}{2}} = 5\frac{1}{2} \text{ hours, and } \frac{2x}{y + 1} = 3\frac{3}{4} \text{ hours.}$$

7. The sum of two numbers which are formed by the same two digits in reverse order is $\frac{4}{5}$ of their difference; and the difference of the squares of the numbers is 3960. Determine the numbers.
8. The hypotenuse of a right triangle is 20, and the area of the triangle is 96. Determine the legs.

NOTE. The square on the hypotenuse = sum of the squares on the legs; and the area of a right triangle = $\frac{1}{2}$ product of legs.

9. Two boys run in opposite directions round a rectangular field the area of which is an acre; they start from one corner and meet 13 yards from the opposite corner; and the rate of one is $\frac{5}{8}$ of the rate of the other. Determine the dimensions of the field.
10. A, in running a race with B to a post and back, met him 10 yards from the post. To make it a dead heat, B must have increased his rate from this point $41\frac{2}{3}$ yards per minute; and if, without changing his pace, he had turned back on meeting A, he would have come in 4 seconds after him. How far was it to the post?
11. The fore wheel of a carriage turns in a mile 132 times more than the hind wheel; but if the circumferences were each increased by 2 feet, it would turn only 88 times more. Find the circumference of each.
12. A person has \$6500, which he divides into two parts and loans at *different rates* of interest, so that the two parts produce *equal* returns. If the first part had been loaned at the second rate of interest, it would have produced \$180; and if the second part had been loaned at the first rate of interest, it would have produced \$245. Find the rates of interest.

CHAPTER XVI.

SIMPLE INDETERMINATE EQUATIONS.

242. If a *single* equation be given which contains *two* unknown numbers, and no other condition be imposed, the number of its solutions is *unlimited*; for, if any value be assigned to one of the unknown numbers, a *corresponding* value may be found for the other. Such an equation is said to be *indeterminate*.

243. The values of the unknown numbers in an indeterminate equation are *dependent upon each other*; so that, though they are unlimited in number, they are confined to a *particular range*.

This range may be still further limited by requiring these values to satisfy some given condition; as, for instance, that they shall be *positive integers*.

244. The method of solving an indeterminate equation in positive integers is as follows:

(1) Solve $3x + 4y = 22$, in positive integers.

Transpose,

$$3x = 22 - 4y,$$

$$\therefore x = 7 - y + \frac{1-y}{3},$$

the quotient being written as a mixed expression.

$$\therefore x + y - 7 = \frac{1-y}{3},$$

Since the values of x and y are to be integral, $x + y - 7$ will be integral, and hence, $\frac{1-y}{3}$ will be integral, though written in the *form* of a fraction.

Let $\frac{1-y}{3} = m$, an integer;

Then $1 - y = 3m,$
 $\therefore y = 1 - 3m.$

Substitute this value of y in the original equation,

$$3x + 4 - 12m = 22,$$

$$\therefore x = 6 + 4m.$$

The equation $y = 1 - 3m$ shows that m in respect to y may be 0, or have any negative value, but cannot have a positive value.

The equation $x = 6 + 4m$ shows that m in respect to x may be 0, but cannot have a negative value greater than 1.

$$\therefore m \text{ may be } 0 \text{ or } -1,$$

and then $x = 6, y = 1;$

or $x = 2, y = 4.$

(2) Solve $5x - 14y = 11$, in positive integers.

Transpose, $5x = 11 + 14y,$
 $\therefore x = 2 + 2y + \frac{1 + 4y}{5},$

$$\therefore x - 2y - 2 = \frac{1 + 4y}{5},$$

$$\therefore \frac{1 + 4y}{5} \text{ must be integral.}$$

Now, if $\frac{1 + 4y}{5}$ be put $= m$, then $y = \frac{5m - 1}{4}$, a fraction in form.

To avoid this difficulty, it is necessary in some way to make the coefficient of y equal to *unity*. Since $\frac{1 + 4y}{5}$ is integral, any multiple of $\frac{1 + 4y}{5}$ is integral. Multiply, then, by such a number as will make the coefficient of y *greater by 1* than some multiple of the denominator. In this case, multiply by 4. Then

$$\frac{4 + 16y}{5} \text{ or } 3y + \frac{4 + y}{5} \text{ is integral.}$$

$$\therefore \frac{4 + y}{5} = m, \text{ an integer;}$$

$$\therefore y = 5m - 4.$$

Since $x = \frac{1}{5}(11 + 14y)$, from the original equation,

$$\therefore x = 14m - 9.$$

Here it is obvious that m may have *any positive value*, and

$$x = 5, 19, 33, \dots$$

$$y = 1, 6, 11, \dots$$

The required multiplier can always be found when the coefficients are prime to each other, and it is best to divide the original equation by the smaller of the two coefficients, in order to have the multiplier as small as possible.

245. The necessity for a multiplier may often be obviated by a little ingenuity. Thus,

The equation $4y = 29 - 7x$ may be put in the form of

$$4y = 29 - 8x + x,$$

$$\therefore y = 7 - 2x + \frac{1+x}{4},$$

in which the fraction is of the required form.

The equation $5x = 18 + 13y$

gives $x = 3 + 2y + \frac{3(1+y)}{5},$

in which $\frac{1+y}{5}$ is of the required form.

246. It will be seen from (1) and (2), § 244, that when only *positive integers* are required, the number of solutions will be *limited* or *unlimited* according as the sign connecting x and y is *positive* or *negative*.

- (1) Find the least number that when divided by 14 and 5 will give remainders 1 and 3 respectively.

If N represents the number, then

$$\frac{N-1}{14} = x, \text{ and } \frac{N-3}{5} = y.$$

$$\therefore N = 14x + 1, \text{ and } N = 5y + 3.$$

$$\therefore 14x + 1 = 5y + 3.$$

$$5y = 14x - 2,$$

$$5y = 15x - 2 - x.$$

$$\therefore y = 3x - \frac{2+x}{5}.$$

Let $\frac{2+x}{5} = m$, an integer;

$$\therefore x = 5m - 2,$$

$$y = \frac{1}{5}(14x - 2), \text{ from original equation.}$$

$$\therefore y = 14m - 6.$$

If $m = 1$,

$$x = 3, \text{ and } y = 8.$$

$$\therefore N = 14x + 1 = 5y + 3 = 43. \quad \text{Ans.}$$

- (2) Solve $5x + 6y = 30$, so that x may be a multiple of y , and both positive.

Let $x = my$.

Then $(5m + 6)y = 30$.

$$\therefore y = \frac{30}{5m + 6},$$

and $x = \frac{30m}{5m + 6}.$

If $m = 2$, $x = 3\frac{1}{2}$, $y = 1\frac{1}{2}$.

If $m = 3$, $x = 4\frac{1}{2}$, $y = 1\frac{1}{2}$.

- (3) Solve $14x + 22y = 71$, in positive integers.

$$x = 5 - y + \frac{1 - 8y}{14},$$

If we multiply the fraction by 7 and reduce,
the result is $-4y + \frac{1}{2}$,

a form which shows there can be no *integral* solution.

There can be no integral solution of $ax \pm by = c$, if a and b have a common factor not common also to c ; for, if d be a factor of a and also of b , but not of c , the equation may be written,

$$mdx \pm ndy = c,$$

or $mx \pm ny = \frac{c}{d}$, a fraction.

EXERCISE 96.

Solve in positive integers :

1. $2x + 11y = 49$.

5. $3x + 8y = 61$.

2. $7x + 3y = 40$.

6. $8x + 5y = 97$.

3. $5x + 7y = 53$.

7. $16x + 7y = 110$.

4. $x + 10y = 29$.

8. $7x + 10y = 206$.

Solve in least positive integers :

9. $12x - 7y = 1$.

12. $23x - 9y = 929$.

10. $5x - 17y = 23$.

13. $23x - 33y = 43$.

11. $23y - 13x = 3$.

14. $555x - 22y = 73$.

15. How many fractions are there with denominators 12 and 18 whose sum is $\frac{3}{4}$?
16. What is the least number which, when divided by 3 and 5, leaves remainders 2 and 3 respectively?
17. A person counting a basket of eggs, which he knows are between 50 and 60, finds that when he counts them 3 at a time there are 2 over; but when he counts them 5 at a time there are 4 over. How many are there in all?
18. A person bought 40 animals, consisting of pigs, geese, and chickens, for \$40. The pigs cost \$5 apiece, the geese \$1, and the chickens 25 cents each. Find the number he bought of each.
19. Find the least multiple of 7 which, when divided by 2, 3, 4, 5, 6, leaves in each case 1 for a remainder.
20. In how many ways may 100 be divided into two parts, one of which shall be a multiple of 7 and the other of 9?
21. Solve $18x - 5y = 70$ so that y may be a multiple of x , and both positive.
22. Solve $8x + 12y = 23$ so that x and y may be positive, and their sum an integer.
23. Divide 70 into three parts which shall give integral quotients when divided by 6, 7, 8, respectively, and the sum of the quotients shall be 10.
24. Divide 200 into three parts which shall give integral quotients when divided by 5, 7, 11, respectively, and the sum of the quotients shall be 20.
25. A number consisting of three digits, of which the middle one is 4, has the digits in the units' and hundreds' places interchanged by adding 792. Find the number.

26. Some men earning each \$2.50 a day, and some women earning each \$1.75 a day, receive all together for their daily wages \$44.75. Determine the number of men and the number of women.
27. A wishes to pay B a debt of £1 12s., but has only half-crowns in his pocket, while B has only 4-penny pieces. How may they settle the matter most simply ?
28. Show that $323x - 527y = 1000$ cannot be satisfied by integral values of x and y .
29. A farmer buys oxen, sheep, and hens. The whole number bought is 100, and the whole price £100. If the oxen cost £5, the sheep £1, and the hens 1s. each, how many of each did he buy ?
30. A number of lengths 3 feet, 5 feet, and 8 feet are cut; how may 48 of them be taken so as to measure 175 feet all together ?
31. A field containing an integral number of acres less than 10 is divided into 8 lots of one size, and 7 of 4 times that size, and has also a road passing through it containing 1300 square yards. Find the size of the lots in square yards.
32. Two wheels are to be made, the circumference of one of which is to be a multiple of the other. What circumferences may be taken so that when the first has gone round three times and the other five, the difference in the length of rope coiled on them may be 17 feet ?
33. In how many ways can a person pay a sum of £15 in half-crowns, shillings, and sixpences, so that the number of shillings and sixpences together shall be equal to the number of half-crowns ?

CHAPTER XVII.

INEQUALITIES.

247. EXPRESSIONS containing any given letter will have their values changed when different values are assigned to that letter; and of two such expressions, one may be for some values of the letter larger than the other, for other values of the letter smaller than the other.

Thus, $1 + x + x^2$ will be greater than $1 - x + x^2$ for all positive values of x , but less for all negative values of x .

248. One expression, however, may be so related to another that, whatever values may be given to the letter, it cannot be greater than the other.

Thus, $2x$ cannot be greater than $x^2 + 1$, whatever value be given to x .

249. For finding whether this relation holds between two expressions, the following is a fundamental proposition:

If a and b are unequal, $a^2 + b^2 > 2ab$.

For, $(a - b)^2$ must be positive, whatever the values of a and b .

That is

$$(a - b)^2 > 0,$$

or

$$a^2 - 2ab + b^2 > 0;$$

$$\therefore a^2 + b^2 > 2ab.$$

250. The principles applied to the solution of equations may be applied to inequalities, except that if each side of an equality have its *sign changed*, the inequality will be *reversed*.

Thus, if $a > b$, then $-a$ will be $< -b$.

- (1) If a and b be positive, show that $a^3 + b^3$ is $> a^2b + ab^2$.

$$\begin{aligned}
 & a^3 + b^3 > a^2b + ab^2, \\
 \text{if (dividing each side by } a + b), & \\
 & a^2 - ab + b^2 > ab, \\
 \text{if} & a^2 + b^2 > 2ab. \\
 \text{But} & a^2 + b^2 \text{ is } > 2ab. \quad \S 249. \\
 & \therefore a^3 + b^3 > a^2b + ab^2.
 \end{aligned}$$

- (2) Show that $a^2 + b^2 + c^2$ is $> ab + ac + bc$.

$$\begin{aligned}
 \text{Now,} & a^2 + b^2 \text{ is } > 2ab, \\
 & a^2 + c^2 \text{ is } > 2ac, \quad \S 249. \\
 & b^2 + c^2 \text{ is } > 2bc. \\
 \text{By adding, } & 2a^2 + 2b^2 + 2c^2 \text{ is } > 2ab + 2ac + 2bc. \\
 & \therefore a^2 + b^2 + c^2 \text{ is } > ab + ac + bc.
 \end{aligned}$$

EXERCISE 97.

Show that, the letters being unequal and positive :

- $a^2 + 3b^2$ is $> 2b(a + b)$.
- $a^3b + ab^3$ is $> 2a^2b^2$.
- $(a^2 + b^2)(a^4 + b^4)$ is $> (a^3 + b^3)^2$.
- $a^2b + a^2c + ab^2 + b^2c + ac^2 + bc^2$ is $> 6abc$.
- The sum of any fraction and its reciprocal is > 2 .
- If $x^2 = a^2 + b^2$, $y^2 = c^2 + d^2$, xy is not $< ac + bd$, or $ad + bc$.
- $ab + ac + bc < (a + b - c)^2 + (a + c - b)^2 + (b + c - a)^2$.
- Which is the greater, $(a^2 + b^2)(c^2 + d^2)$ or $(ac + bd)^2$?
- Which is the greater, $m^2 + m$ or $m^3 + 1$?
- Which is the greater, $a^4 - b^4$ or $4a^3(a - b)$ when a is $> b$?
- Which is the greater, $\sqrt{\frac{a^2}{b}} + \sqrt{\frac{b^2}{a}}$ or $\sqrt{a} + \sqrt{b}$?
- Which is the greater, $\frac{a+b}{2}$ or $\frac{2ab}{a+b}$?
- Which is the greater, $\frac{a}{b^2} + \frac{b}{a^2}$ or $\frac{1}{b} + \frac{1}{a}$?

CHAPTER XVIII.

THEORY OF EXPONENTS.

251. THE expression a^n , when n is a positive integer, has been defined as the product of n equal factors each equal to a . § 24.

And it has been shown that $a^m \times a^n = a^{m+n}$. § 66.

That $a^m \div a^n = a^{m-n}$, if m is greater than n ; § 93.

or $\frac{1}{a^{n-m}}$, if m is less than n . § 94.

And that $(a^m)^n = a^{mn}$. § 199.

Also, it is true that $a^n \times b^n = (ab)^n$; for
 $(ab)^n = ab$ taken n times as a factor,
 $= a$ taken n times as a factor $\times b$ taken n times as a factor
 $= a^n \times b^n$.

252. Likewise, $\sqrt[n]{a}$, when n is a positive integer, has been defined as *one of the n equal factors of a* (§ 203); so that if $\sqrt[n]{a}$ be taken n times as a factor, the resulting product is a ; that is, $(\sqrt[n]{a})^n = a$.

Again, the expression $\sqrt[n]{a^m}$ means that a is to be raised to the m th power, and the n th root of the result obtained.

And the expression $(\sqrt[n]{a})^m$ means that the n th root of a is to be taken, and the result raised to the m th power.

It will thus be seen that any proposition relating to roots and powers may be expressed by this method of notation. It is, however, *found convenient* to adopt another method of notation, in which fractional and negative exponents are used.

253. The meaning of a fractional exponent is at once suggested, by observing that the *division of an exponent*, when the resulting quotient is *integral*, is equivalent to extracting a root. Thus, a^3 is the square root of a^6 , and 3, the exponent of a^3 , is obtained by dividing the exponent of a^6 by 2.

If this division be indicated only, the square root of a^6 will be denoted by $a^{\frac{6}{2}}$, in which the *denominator* denotes the *root*, and the *numerator* the *power*. If the same meaning be given to an exponent when the division does not give an integral quotient, $a^{\frac{3}{2}}$ will represent the square root of the cube of a ; and, in general, $a^{\frac{m}{n}}$, the n th root of the m th power of a . This, then, is the meaning that will be assigned to a fractional exponent, so that in a fractional exponent

254. *The numerator will indicate a power, and the denominator a root.*

255. The meaning of a negative exponent is suggested by observing that in a series of descending powers of a ,

$$a^n \dots a^5, a^4, a^3, a^2, a^1,$$

the subtraction of 1 from the exponent is equivalent to dividing by a ; and if the operation be continued, the result is

$$a^0, a^{-1}, a^{-2}, a^{-3}, a^{-4} \dots a^{-n}.$$

Then

$$a^0 = \frac{a}{a} = 1; \quad a^{-1} = 1 \div a = \frac{1}{a};$$

$$a^{-2} = \frac{1}{a} \div a = \frac{1}{a^2}; \quad a^{-n} = \frac{1}{a^n}.$$

This, then, is the meaning that will be assigned to a negative exponent, so that,

256. *A number with a negative exponent will denote the reciprocal of the number with the corresponding positive exponent.*

It may be easily shown that the laws which apply to positive integral exponents apply also to fractional and negative exponents.

257. To show that $a^{\frac{m}{n}} \times b^{\frac{m}{n}} = (ab)^{\frac{m}{n}}$:

$$\begin{aligned} a^{\frac{m}{n}} \times b^{\frac{m}{n}} &= \sqrt[n]{a^m} \times \sqrt[n]{b^m}, \\ &= \sqrt[n]{a^m b^m}, \\ &= \sqrt[n]{(ab)^m}, \\ &= (ab)^{\frac{m}{n}} \quad (\text{by definition}). \end{aligned}$$

Likewise $a^{\frac{1}{n}} \times b^{\frac{1}{n}} \times c^{\frac{1}{n}} = (abc)^{\frac{1}{n}}$, and so on.

258. To show that $(a^{\frac{1}{m}})^{\frac{1}{n}} = a^{\frac{1}{mn}}$:

Let $x = (a^{\frac{1}{m}})^{\frac{1}{n}}.$

Then $x^n = a^{\frac{1}{m}},$ and $x^{mn} = a.$

$$\therefore x = a^{\frac{1}{mn}}.$$

But $x = (a^{\frac{1}{m}})^{\frac{1}{n}} \quad (\text{by supposition}).$

$$\therefore (a^{\frac{1}{m}})^{\frac{1}{n}} = a^{\frac{1}{mn}}.$$

259. To show that $a^m \times a^{-n} = a^{m-n}$:

Now $a^m \times a^{-n} = a^m \times \frac{1}{a^n},$

$$= \frac{a^m}{a^n} = a^{m-n} \text{ if } m > n; \quad \S 93.$$

or $= \frac{1}{a^{n-m}} \text{ if } m < n, \quad \S 94.$

$$\begin{aligned} &= a^{-(n-m)} \quad (\text{by definition}), \\ &= a^{m-n}. \end{aligned}$$

260. In like manner the same laws may be shown to apply in every case.

261. Hence, whether m and n are *integral* or *fractional*, *positive* or *negative*:

$$\text{I. } a^m \times a^n = a^{m+n}.$$

$$\text{III. } (a^m)^n = a^{mn}.$$

$$\text{II. } a^m \div a^n = a^{m-n}.$$

$$\text{IV. } a^m \times b^m = (ab)^m.$$

EXERCISE 98.

Express with fractional exponents:

$$1. \sqrt{x^3}; \sqrt[3]{x^2}; (\sqrt{x})^5; \sqrt[3]{a^4}; \sqrt[7]{a^6}; (\sqrt[3]{a})^7; \sqrt[6]{a^3b^2}.$$

$$2. \sqrt[3]{xy^2z^3}; \sqrt[5]{x^3y^2z^4}; \sqrt[7]{a^5b^6c^7}; 5\sqrt{a^2bc^3x^4}.$$

Express with radical signs:

$$3. a^{\frac{1}{2}}; a^{\frac{1}{3}}b^{\frac{1}{4}}; 4x^{\frac{1}{2}}y^{-\frac{1}{3}}; 3x^{\frac{1}{2}}y^{-\frac{1}{3}}.$$

Express with positive exponents:

$$4. a^{-2}; 3x^{-1}y^{-3}; 6x^{-3}y; x^4y^{-5}; \frac{2a^{-1}x}{3^{-1}b^2y^{-3}}.$$

Write in the form of integral expressions:

$$5. \frac{3xy}{z^2}; \frac{z}{x^3y^4}; \frac{a}{bc}; \frac{c^2}{a^3b^{-2}}; \frac{x^{-\frac{1}{2}}}{y^{-\frac{1}{3}}}; \frac{x^{-2}}{y^{\frac{1}{2}}}.$$

Simplify:

$$6. a^{\frac{1}{2}} \times a^{\frac{1}{3}}; b^{\frac{1}{2}} \times b^{\frac{1}{3}}; c^{\frac{1}{2}} \times c^{\frac{1}{3}}; d^{\frac{1}{2}} \times d^{\frac{1}{3}}.$$

$$7. m^{\frac{1}{2}} \times m^{-\frac{1}{3}}; n^{\frac{1}{2}} \times n^{-\frac{1}{3}}; a^0 \times a^{\frac{1}{2}}; a^0 \times a^{-\frac{1}{2}}.$$

$$8. a^{\frac{1}{2}} \times \sqrt{a}; c^{-\frac{1}{2}} \times \sqrt{c}; y^{\frac{1}{2}} \times \sqrt[4]{y}; x^{\frac{1}{2}} \times \sqrt{x^{-1}}.$$

$$9. ab^{\frac{1}{2}}c \times a^{-\frac{1}{2}}bc^{\frac{1}{2}}; a^{\frac{1}{2}}b^{\frac{1}{2}}c^{-\frac{1}{2}} \times a^{\frac{1}{2}}b^{-\frac{1}{2}}cd.$$

$$10. x^{\frac{1}{2}}y^{\frac{1}{3}}z^{\frac{1}{4}} \times x^{-\frac{1}{2}}y^{-\frac{1}{3}}z^{-\frac{1}{4}}; x^{\frac{1}{2}}y^{\frac{1}{3}}z^{\frac{1}{4}} \times x^{-\frac{1}{2}}y^{-\frac{1}{3}}z^{-\frac{1}{4}}.$$

$$11. a^{\frac{1}{2}} \times a^{-\frac{1}{3}} \times a^{-\frac{1}{4}} \times a^{-\frac{1}{5}}; \left(\frac{ay}{x}\right)^{\frac{1}{2}} \times \left(\frac{bx}{y^2}\right)^{\frac{1}{3}} \times \left(\frac{y^2}{a^2b^2}\right)^{\frac{1}{4}}.$$

12. $a^{\frac{1}{2}} \div a^{\frac{1}{2}}; c^{\frac{1}{2}} \div c^{\frac{1}{2}}; n^{\frac{1}{2}} \div n^{\frac{1}{2}}; a^{\frac{1}{2}} \div \sqrt[3]{a^{\frac{1}{2}}}$.
 13. $(a^6)^{\frac{1}{2}} \div (a^6)^{\frac{1}{2}}; (c^{-\frac{1}{2}})^{\frac{1}{2}}; (m^{-\frac{1}{2}})^{\frac{1}{2}}; (n^{\frac{1}{2}})^{-3}; (x^{\frac{1}{2}})^{\frac{1}{2}}$.
 14. $(p^{-\frac{1}{2}})^{-\frac{1}{2}}; (q^{\frac{1}{2}})^{-\frac{1}{2}}; (x^{-\frac{1}{2}}y^{\frac{1}{2}})^{-\frac{1}{2}}; (a^{\frac{1}{2}} \times a^{\frac{1}{2}})^{-\frac{1}{2}}$.
 15. $(4a^{-\frac{1}{2}})^{-\frac{1}{2}}; (27b^{-\frac{1}{2}})^{-\frac{1}{2}}; (64c^{16})^{-\frac{1}{2}}; (32c^{-10})^{\frac{1}{2}}$.
 16. $\left(\frac{16a^{-4}}{81b^3}\right)^{-\frac{1}{2}}; \left(\frac{9a^4}{16b^{-3}}\right)^{-\frac{1}{2}}; (3^{\frac{1}{2}}a^{-5})^{-\frac{1}{2}}; (\frac{2}{3}\frac{4}{5}\frac{6}{7})^{-\frac{1}{2}}$.

262. The laws that apply to the exponents of simple expressions also apply to the exponents of compound expressions.

- (1) Multiply $y^{\frac{1}{2}} + y^{\frac{1}{2}} + y^{\frac{1}{2}} + 1$ by $y^{\frac{1}{2}} - 1$.

$$\begin{array}{r} y^{\frac{1}{2}} + y^{\frac{1}{2}} + y^{\frac{1}{2}} + 1 \\ y^{\frac{1}{2}} - 1 \\ \hline y + y^{\frac{1}{2}} + y^{\frac{1}{2}} + y^{\frac{1}{2}} \\ - y^{\frac{1}{2}} - y^{\frac{1}{2}} - y^{\frac{1}{2}} - 1 \\ \hline y \qquad \qquad \qquad -1 \end{array} \qquad y - 1. \text{ Ans.}$$

- (2) Divide $x^{\frac{1}{2}} + x^{\frac{1}{2}} - 12$ by $x^{\frac{1}{2}} - 3$.

$$\begin{array}{r} x^{\frac{1}{2}} + x^{\frac{1}{2}} - 12 \left(\frac{x^{\frac{1}{2}} - 3}{x^{\frac{1}{2}} + 4} \right) \\ x^{\frac{1}{2}} - 3x^{\frac{1}{2}} \\ \hline 4x^{\frac{1}{2}} - 12 \\ \underline{4x^{\frac{1}{2}} - 12} \end{array} \qquad x^{\frac{1}{2}} + 4. \text{ Ans.}$$

EXERCISE 99.

Multiply:

- $x^{2p} + x^p y^p + y^{2p}$ by $x^{2p} - x^p y^p + y^{2p}$.
- $x^{m-n} - y^n$ by $x^n + y^{m-n}$.
- $x^{\frac{1}{2}} - 2x^{\frac{1}{2}} + 1$ by $x^{\frac{1}{2}} - 1$.
- $8a^{\frac{1}{2}} + 4a^{\frac{1}{2}}b^{\frac{1}{2}} + 5a^{\frac{1}{2}}b^{\frac{1}{2}} + 9b^{\frac{1}{2}}$ by $2a^{\frac{1}{2}} - b^{\frac{1}{2}}$.

5. $1 + ab^{-1} + a^2b^{-2}$ by $1 - ab^{-1} + a^2b^{-2}$.
 6. $a^2b^{-2} + 2 + a^{-2}b^2$ by $a^2b^{-2} - 2 - a^{-2}b^2$.
 7. $4x^{-3} + 3x^{-2} + 2x^{-1} + 1$ by $x^{-2} - x^{-1} + 1$.

Divide:

8. $x^{4n} - y^{4n}$ by $x^n - y^n$.
 9. $x + y + z - 3x^{\frac{1}{2}}y^{\frac{1}{2}}z^{\frac{1}{2}}$ by $x^{\frac{1}{2}} + y^{\frac{1}{2}} + z^{\frac{1}{2}}$.
 10. $x + y$ by $x^{\frac{1}{2}} - x^{\frac{1}{2}}y^{\frac{1}{2}} + x^{\frac{1}{2}}y^{\frac{1}{2}} - x^{\frac{1}{2}}y^{\frac{1}{2}} + y^{\frac{1}{2}}$.
 11. $x^2y^{-2} + 2 + x^{-2}y^2$ by $xy^{-1} + x^{-1}y$.
 12. $a^{-4} + a^{-2}b^{-2} + b^{-4}$ by $a^{-2} - a^{-1}b^{-1} + b^{-2}$.

Find the squares of:

13. $4ab^{-1}$; $a^{\frac{1}{2}} - b^{\frac{1}{2}}$; $a + a^{-1}$; $2a^{\frac{1}{2}}b^{\frac{1}{2}} - a^{-\frac{1}{2}}b^{\frac{1}{2}}$.

If $a=4$, $b=2$, $c=1$, find the values of:

14. $a^{\frac{1}{2}}b$; $5ab^{-1}$; $2(ab)^{\frac{1}{2}}$; $a^{-\frac{1}{2}}b^{-1}c^{\frac{1}{2}}$; $12a^{-2}b^{-3}$.
 15. Expand $(a^{\frac{1}{2}} - b^{\frac{1}{2}})^3$; $(2x^{-1} + x)^4$; $(ab^{-1} - by^{-1})^6$.

Extract the square root of:

16. $9x^{-4} - 18x^{-3}y^{\frac{1}{2}} + 15x^{-2}y - 6x^{-1}y^{\frac{3}{2}} + y^2$.

Extract the cube root of:

17. $8x^3 + 12x^2 - 30x - 35 + 45x^{-1} + 27x^{-2} - 27x^{-3}$.

Resolve into prime factors with fractional exponents:

18. $\sqrt[3]{12}$, $\sqrt[4]{72}$, $\sqrt[6]{96}$, $\sqrt[8]{64}$; and find their product.

Simplify:

19. $\{(x^{ab})^3 \times (x^{cb})^{-2}\}^{\frac{1}{3a-2}}$. 20. $(x^{18a} \times x^{-12})^{\frac{1}{3a-2}}$.
 21. $3(a^{\frac{1}{2}} + b^{\frac{1}{2}})^2 - 4(a^{\frac{1}{2}} + b^{\frac{1}{2}})(a^{\frac{1}{2}} - b^{\frac{1}{2}}) + (a^{\frac{1}{2}} - 2b^{\frac{1}{2}})^2$.
 22. $\{(a^m)^{m-\frac{1}{m}}\}^{\frac{1}{m+1}}$. 2A. $[\{(a^{-m})^{-n}\}^p]^q \div [\{(a^m)^n\}^{-p}]^{-q}$.
 23. $\left(\frac{x^{p+q}}{x^q}\right)^p \div \left(\frac{x^q}{x^{q-p}}\right)^{p-q}$. 25. $\frac{x^{2p(q-1)} - y^{2q(p-1)}}{x^{p(q-1)} + y^{q(p-1)}}$.

RADICAL EXPRESSIONS.

263. An indicated root that cannot be exactly obtained is called a **surd**. An indicated root that can be exactly obtained is said to have the *form* of a surd.

264. The required root shows the **order** of a surd; and surds are named *quadratic*, *cubic*, *biquadratic*, according as the *second*, *third*, or *fourth* roots are required.

265. The product of a rational factor and a surd factor is called a *mixed surd*; as, $3\sqrt{2}$, $b\sqrt{a}$.

266. When there is no rational factor outside of the radical sign, the surd is said to be *entire*; as, $\sqrt{2}$, \sqrt{a} .

267. Since $\sqrt[n]{a} \times \sqrt[n]{b} \times \sqrt[n]{c} = \sqrt[n]{abc}$, the product of two or more surds of the same order will be a radical expression of the same order consisting of the product of the numbers under the radical signs.

268. In like manner, $\sqrt{a^2b} = \sqrt{a^2} \times \sqrt{b} = a\sqrt{b}$. That is,
A factor under the radical sign whose root can be taken, may, by having the root taken, be removed from under the radical sign.

269. Conversely, since $a\sqrt{b} = \sqrt{a^2b}$,
A factor outside the radical sign may be raised to the corresponding power and placed under it.

Again:
$$\sqrt{\frac{a}{b^2}} = \sqrt{a \times \frac{1}{b^2}} = \frac{1}{b} \sqrt{a};$$

and
$$\sqrt{\frac{a}{b}} = \sqrt{\frac{ab}{b^2}} = \sqrt{ab \times \frac{1}{b^2}} = \frac{1}{b} \sqrt{ab}.$$

270. A surd is in its *simplest form* when the expression under the radical sign is *integral* and *as small as possible*.

271. Surds which, when reduced to the simplest form, have the *same surd factor*, are said to be similar.

Simplify :

$$\sqrt{50}; \quad \sqrt[3]{108}; \quad \sqrt[5]{7x^2y^7}; \quad \sqrt{\frac{7}{12}}; \quad \sqrt[4]{\frac{5a}{2b^3c^3}}; \quad \sqrt[3]{296352}.$$

$$(1) \sqrt{50} = \sqrt{25 \times 2} = 5\sqrt{2}.$$

$$(2) \sqrt[3]{108} = \sqrt[3]{27 \times 4} = 3\sqrt[3]{4}.$$

$$(3) \sqrt[5]{7x^2y^7} = \sqrt[5]{7x^2y^2 \times y^5} = y\sqrt[5]{7x^2y^2}.$$

$$(4) \sqrt{\frac{7}{12}} = \sqrt{\frac{7}{4 \times 3}} = \sqrt{\frac{7 \times 3}{4 \times 9}} = \sqrt{21 \times \frac{1}{4 \times 9}} = \frac{1}{6}\sqrt{21}.$$

$$(5) \sqrt[4]{\frac{5a}{2b^3c^3}} = \sqrt[4]{\frac{40abc^2}{16b^4c^4}} = \frac{1}{2bc}\sqrt[4]{40abc^2}.$$

$$(6) \sqrt[3]{296352}.$$

2 ³	296352
2 ²	37044
3 ²	9261
3	1029
7	343
7	49
7	

$$\text{Hence, } 296352 = 2^8 \times 3^3 \times 7^3,$$

$$\begin{aligned} \therefore \sqrt[3]{296352} &= \sqrt[3]{7^3} \times \sqrt[3]{3^3} \times \sqrt[3]{2^6}, \\ &= 7 \times 3 \times 2\sqrt[3]{2^2}, \\ &= 42\sqrt[3]{4}. \text{ Ans.} \end{aligned}$$

In simplifying numerical expressions under the radical sign, the method employed in (6) may be used with advantage when the factor whose root can be taken is not readily determined by inspection.

EXERCISE 100.

Express as entire surds :

- $3\sqrt{5}; 3\sqrt{21}; 5\sqrt{32}; a^2b\sqrt{bc}; x\sqrt{x^2y^2}.$
- $3y^2\sqrt[4]{x^3y}; 2x\sqrt[5]{xy}; a^3\sqrt[4]{a^3b^2}; 3c^2\sqrt[3]{abc}; 5abc\sqrt{abc^{-1}}.$
- $4\sqrt[3]{21}; 16\sqrt[4]{114}; (x+y)\sqrt{\frac{xy}{x^2+2xy+y^2}}.$

Express as mixed surds :

4. $\sqrt{x^2 y^4 z}$; $\sqrt{8 a^3 b}$; $\sqrt[3]{54 a^4 x^2 y^3}$; $\sqrt{24}$; $\sqrt{125 a^4 d^3}$.
5. $\sqrt[3]{1000 a}$; $\sqrt[3]{160 x^4 y^7}$; $\sqrt[3]{108 m^9 n^{10}}$; $\sqrt[3]{1372 a^{15} b^{16}}$.
6. $\sqrt[3]{a^4 - 3 a^3 b + 3 a^2 b^2 - a b^3}$; $\sqrt{50 a^2 - 100 a b + 50 b^2}$.

Simplify :

7. $2\sqrt[4]{80 a^5 b^2 c^6}$; $7\sqrt{396 x}$; $9\sqrt[3]{81 x^2 y^3 z}$; $5\sqrt{726}$.
8. $\sqrt{\frac{2}{3}}$; $\sqrt{1\frac{1}{8}}$; $\sqrt{3\frac{1}{8}}$; $\frac{2}{3}\sqrt{90\frac{2}{3}}$; $2\sqrt[3]{\frac{1}{2}}$.
9. $\sqrt[3]{\frac{2xy^2}{z}}$; $\sqrt[3]{\frac{4}{25}}$; $\frac{a}{b}\sqrt[4]{\frac{b}{2a^3}}$; $\sqrt{\frac{3a^2bx}{4cy^3}}$.
10. $\frac{12}{\sqrt{5}}$; $\frac{2}{\sqrt{1701}}$; $\left(\frac{x^3 y^2}{z^2}\right)\left(\frac{z^5}{x^5 y^5}\right)^{\frac{1}{2}}$; $\left(\frac{a^3 b^3}{c^4}\right)\left(\frac{c^9 b^3}{a}\right)^{\frac{1}{2}}$.
11. $(ax) \times (b^2 x)^{\frac{1}{2}}$; $(2 a^2 b^4) \times (b^2 x^3)^{\frac{1}{2}}$; $5(3 a^3 b^4 y) \times (a^5 b^{-4} y^3)^{\frac{1}{2}}$.
12. Show that $\sqrt{20}$, $\sqrt{45}$, $\sqrt{\frac{1}{3}}$ are similar surds.
13. Show that $2\sqrt[3]{a^3 b^2}$, $\sqrt[3]{8 b^5}$, $\frac{1}{2}\sqrt[3]{\frac{a^6}{b}}$ are similar surds.
14. If $\sqrt{2} = 1.414213$, find the values of
 $\sqrt{50}$; $\frac{2}{3}\sqrt{288}$; $\frac{1}{\sqrt{2}}$; $\frac{3}{\sqrt{450}}$.

272. Surds of the same order may be compared by expressing them as entire surds.

Ex. Compare $\frac{2}{3}\sqrt{7}$ and $\frac{3}{5}\sqrt{10}$.

$$\frac{2}{3}\sqrt{7} = \sqrt{\frac{28}{9}},$$

$$\frac{3}{5}\sqrt{10} = \sqrt{\frac{18}{5}}.$$

$$\sqrt{\frac{28}{9}} = \sqrt{\frac{140}{45}}, \text{ and } \sqrt{\frac{18}{5}} = \sqrt{\frac{162}{45}}.$$

As $\sqrt{\frac{140}{45}}$ is greater than $\sqrt{\frac{162}{45}}$, $\frac{2}{3}\sqrt{7}$ is greater than $\frac{3}{5}\sqrt{10}$.

273. The product or quotient of two surds of *the same order* may be obtained by taking the product or quotient of the rational factors and the surd factors separately.

$$(1) \quad 2\sqrt{5} \times 5\sqrt{7} = 10\sqrt{35}.$$

$$(2) \quad 9\sqrt{5} \div 3\sqrt{7} = 3\sqrt{\frac{5}{7}} = 3\sqrt{\frac{35}{49}} = \frac{3}{7}\sqrt{35}.$$

EXERCISE 101.

1. Which is the greater $3\sqrt{7}$ or $2\sqrt{15}$?
2. Arrange in order of magnitude $9\sqrt{3}$, $6\sqrt{7}$, $5\sqrt{10}$.
3. Arrange in order of magnitude $4\sqrt[3]{4}$, $3\sqrt[3]{5}$, $5\sqrt[3]{3}$.
4. Multiply $3\sqrt{2}$ by $4\sqrt{6}$; $\frac{2}{3}\sqrt{10}$ by $\frac{1}{10}\sqrt{15}$.
5. Multiply $5\sqrt{\frac{1}{2}}$ by $\frac{2}{3}\sqrt{162}$; $\frac{1}{2}\sqrt[3]{4}$ by $2\sqrt[3]{2}$.
6. Divide $2\sqrt{5}$ by $3\sqrt{15}$; $\frac{8}{9}\sqrt{21}$ by $\frac{2}{15}\sqrt{\frac{7}{5}}$.
7. Simplify $\frac{2}{3}\sqrt{3} \times \frac{4}{5}\sqrt{5} \div \frac{1}{6}\sqrt{2}$.
8. Simplify $\frac{2\sqrt{10}}{3\sqrt{27}} \times \frac{7\sqrt{48}}{5\sqrt{14}} \div \frac{4\sqrt{15}}{15\sqrt{21}}$.
9. Simplify $2\sqrt[3]{4} \times 5\sqrt[3]{32} \div \sqrt[3]{108}$.

274. The *order* of a surd may be changed by changing the *power* of the expression under the radical sign. Thus,

$$\sqrt{5} = \sqrt[4]{25}; \quad \sqrt[3]{c} = \sqrt[6]{c^2}.$$

$$\text{Conversely,} \quad \sqrt[4]{25} = \sqrt{5}; \quad \sqrt[6]{c^2} = \sqrt[3]{c};$$

$$\text{or, in general,} \quad \sqrt[m]{c^n} = \sqrt[n]{c^m}.$$

In this way, surds of *different orders* may be reduced to the *same order*, and may then be compared, multiplied, or divided.

(1) To compare $\sqrt{2}$ and $\sqrt[3]{3}$.

$$\sqrt{2} = 2^{\frac{1}{2}} = 2^{\frac{3}{6}} = \sqrt[6]{2^3} = \sqrt[6]{8};$$

$$\sqrt[3]{3} = 3^{\frac{1}{3}} = 3^{\frac{2}{6}} = \sqrt[6]{3^2} = \sqrt[6]{9}.$$

$\therefore \sqrt[3]{3}$ is greater than $\sqrt{2}$.

(2) To multiply $\sqrt[3]{4a}$ by $\sqrt{6x}$.

$$\sqrt[3]{4a} = (4a)^{\frac{1}{3}} = (4a)^{\frac{2}{6}} = \sqrt[6]{(4a)^2} = \sqrt[6]{16a^2};$$

$$\sqrt{6x} = (6x)^{\frac{1}{2}} = (6x)^{\frac{3}{6}} = \sqrt[6]{(6x)^3} = \sqrt[6]{216x^3}.$$

$$\therefore \sqrt[3]{4a} \times \sqrt{6x} = \sqrt[6]{16a^2} \times \sqrt[6]{216x^3},$$

$$= \sqrt[6]{16a^2 \times 216x^3},$$

$$= \sqrt[6]{2^4 a^2 \times 2^3 \times 3^3 x^3}.$$

$$= \sqrt[6]{2^6 \times 2 \times 3^3 a^2 x^3},$$

$$= 2\sqrt[6]{54 a^2 x^3}. \text{ Ans.}$$

(3) To divide $\sqrt[3]{3a}$ by $\sqrt{6b}$.

$$\sqrt[3]{3a} = (3a)^{\frac{1}{3}} = (3a)^{\frac{2}{6}} = \sqrt[6]{(3a)^2} = \sqrt[6]{9a^2};$$

$$\sqrt{6b} = (6b)^{\frac{1}{2}} = (6b)^{\frac{3}{6}} = \sqrt[6]{(6b)^3} = \sqrt[6]{216b^3}.$$

$$\therefore \sqrt[3]{3a} \div \sqrt{6b} = \sqrt[6]{9a^2} \div \sqrt[6]{216b^3} = \sqrt[6]{\frac{9a^2}{216b^3}},$$

$$= \sqrt[6]{\frac{a^2}{24b^3}} = \sqrt[6]{\frac{a^2}{2^3 \times 3b^3}},$$

$$= \sqrt[6]{\frac{2^3 \times 3^5 a^2 b^3}{2^6 \times 3^6 b^3}} = \frac{1}{6b} \sqrt[6]{1944 a^2 b^3}. \text{ Ans.}$$

EXERCISE 102.

Arrange in order of magnitude :

1. $2\sqrt[3]{3}, 3\sqrt{2}, \frac{4}{5}\sqrt[4]{4}.$

3. $2\sqrt[3]{22}, 3\sqrt[3]{7}, 4\sqrt{2}.$

2. $\sqrt{\frac{3}{2}}, \sqrt[3]{\frac{1}{2}}.$

4. $3\sqrt{19}, 5\sqrt[3]{2}, 3\sqrt[3]{3}.$

Simplify :

5. $2\sqrt{ax} \times \sqrt[3]{3a^2b} \times \sqrt{2bx} ; \sqrt[4]{a^3xy^2} \times \sqrt[5]{a^2xy}.$
6. $3(4ab^2)^{\frac{1}{2}} \div (2a^3b)^{\frac{1}{2}} ; (2a^3b^2)^{\frac{1}{2}} \times (a^2b^3)^{\frac{1}{2}} \div (a^3b^5)^{\frac{1}{2}}.$
7. $(2ab)^{\frac{1}{2}} \times (3ab^3)^{\frac{1}{2}} \div (5ab^5)^{\frac{1}{2}} ; 4\sqrt{12} \div 2\sqrt{3}.$
8. $\left(\frac{ay}{x}\right)^{\frac{1}{2}} \times \left(\frac{bx}{y^2}\right)^{\frac{1}{2}} \div \left(\frac{y^2}{a^2b^2}\right)^{\frac{1}{2}}.$
9. $(7\sqrt{2} - 5\sqrt{6} - 3\sqrt{8} + 4\sqrt{20}) \times 3\sqrt{2}.$
10. $\sqrt{\left(\frac{1}{2}\frac{2}{3}\right)^7} \times \sqrt{\left(\frac{2}{3}\frac{3}{4}\right)^6} ; \sqrt[3]{(4ab^2)^x} \times \sqrt[3]{(2a^2b)^x}.$
11. $(\sqrt[7]{a^3b})^3 \times (\sqrt[7]{a^3b^{12}})^4 ; a^{\frac{1}{2}}b^{-\frac{1}{3}}c^{\frac{2}{3}}d^{-\frac{1}{6}} \div a^{\frac{1}{2}}b^{-\frac{1}{3}}c^{-\frac{1}{6}}d^{\frac{1}{6}}.$

275. In the addition or subtraction of surds, each surd must be reduced to its simplest form ; and, if the resulting surds are similar,

Add the rational factors, and to their sum annex the common surd factor.

If the resulting surds are not similar,

Connect them with their proper signs.

276. Operations with surds will be more easily performed if the arithmetical numbers contained in the surds are *expressed in their prime factors*, and if *fractional exponents* are used instead of radical signs.

(1) Simplify $\sqrt{27} + \sqrt{48} + \sqrt{147}.$

$$\sqrt{27} = (3^3)^{\frac{1}{2}} = 3 \times 3^{\frac{1}{2}} = 3\sqrt{3} ;$$

$$\sqrt{48} = (2^4 \times 3)^{\frac{1}{2}} = 2^2 \times 3^{\frac{1}{2}} = 4 \times 3^{\frac{1}{2}} = 4\sqrt{3} ;$$

$$\sqrt{147} = (7^2 \times 3)^{\frac{1}{2}} = 7 \times 3^{\frac{1}{2}} = 7\sqrt{3}.$$

$$\therefore \sqrt{27} + \sqrt{48} + \sqrt{147} = (3 + 4 + 7)\sqrt{3} = 14\sqrt{3}. \text{ Ans.}$$

(2) Simplify $2\sqrt[3]{320} - 3\sqrt[3]{40}$.

$$2\sqrt[3]{320} = 2(2^6 \times 5)^{\frac{1}{3}} = 2 \times 2^2 \times 5^{\frac{1}{3}} = 8\sqrt[3]{5};$$

$$3\sqrt[3]{40} = 3(2^3 \times 5)^{\frac{1}{3}} = 3 \times 2 \times 5^{\frac{1}{3}} = 6\sqrt[3]{5}$$

$$\therefore 2\sqrt[3]{320} - 3\sqrt[3]{40} = 8\sqrt[3]{5} - 6\sqrt[3]{5} = 2\sqrt[3]{5}. \text{ Ans.}$$

(3) Find the square root of $\sqrt[3]{81}$.

$$\begin{aligned} \text{The square root of } \sqrt[3]{81} &= (81^{\frac{1}{3}})^{\frac{1}{2}} = 81^{\frac{1}{6}} = (3^4)^{\frac{1}{6}} \\ &= 3^{\frac{2}{3}} = (3^2)^{\frac{1}{3}} = \sqrt[3]{9}. \end{aligned}$$

(4) Find the cube of $\frac{1}{2}\sqrt[6]{2}$.

$$\text{The cube of } \frac{1}{2}\sqrt[6]{2} = \left(\frac{1}{2}\right)^3 \times (2^{\frac{1}{6}})^3 = \frac{1}{8} \times 2^{\frac{1}{2}} = \frac{1}{8}\sqrt{2}.$$

EXERCISE 103.

Simplify :

- $\sqrt{27} + 2\sqrt{48} + 3\sqrt{108}; 3\sqrt{1000} + 4\sqrt{50} + 12\sqrt{288}.$
- $\sqrt[3]{128} + \sqrt[3]{686} + \sqrt[3]{16}; 7\sqrt[3]{54} + 3\sqrt[3]{16} + \sqrt[3]{432}.$
- $12\sqrt{72} - 3\sqrt{128}; 7\sqrt[3]{81} - 3\sqrt[3]{1029}.$
- $2\sqrt{3} + 3\sqrt{1\frac{1}{3}} - \sqrt{5\frac{1}{3}}; 2\sqrt{\frac{5}{3}} + \sqrt{60} - \sqrt{15} - \sqrt{\frac{3}{5}}.$
- $\sqrt{\frac{a^4c}{b^3}} - \sqrt{\frac{a^2c^3}{bd^2}} - \sqrt{\frac{a^2cd^2}{bm^2}}; 3\sqrt{\frac{2}{3}} + 2\sqrt{\frac{1}{16}} - 4\sqrt{\frac{1}{40}}.$
- $\sqrt{4a^3b} + \sqrt{25ab^3} - (a - 5b)\sqrt{ab}.$
- $c\sqrt[5]{a^8b^7c^3} - a\sqrt[5]{ab^7c^3} + b\sqrt[5]{a^8b^2c^3}.$
- $2\sqrt[3]{40} + 3\sqrt[3]{108} + \sqrt[3]{500} - \sqrt[3]{320} - 2\sqrt[3]{1372}.$
- $(2\sqrt[6]{3a^4b})^3; (3\sqrt[6]{3})^2.$
- $\left(\frac{a}{3}\sqrt[4]{\frac{a}{3}}\right)^{\frac{1}{2}}; (\sqrt{27})^{\frac{1}{2}}.$
- $(\sqrt[3]{81})^{\frac{1}{2}}; (\sqrt[4]{512})^{\frac{1}{2}}; (\sqrt[3]{256})^{-\frac{1}{2}}; \sqrt[12]{16}; \sqrt[12]{27}.$
- $\sqrt[10]{4}; \sqrt[10]{36}; \sqrt[10]{32}; \sqrt[10]{243}; \sqrt[9]{125}; \sqrt[4]{49}.$
- $\sqrt[9]{8x^6}; \sqrt[6]{9a^2b^4}; \sqrt[8]{16a^{12}}; \sqrt[5]{32a^{10}}.$

$$14. (\sqrt[9]{8})^4; (\sqrt[9]{27})^4; (\sqrt[8]{64})^3; (\sqrt[3]{4})^2.$$

$$15. (a\sqrt[3]{a})^{-3}; (x\sqrt[3]{x})^{-\frac{1}{2}}; (p^2\sqrt{p})^{\frac{1}{2}}; (a^{-3}\sqrt[4]{a^{-3}})^{-\frac{1}{2}}.$$

Expand by the method explained in § 201 :

$$16. (\sqrt{a} + \sqrt{b})^5; (\sqrt[3]{m^2} + \sqrt{x^3})^3; (\sqrt{a} - 2\sqrt{b})^5.$$

$$17. (2a^2 - \frac{1}{2}\sqrt{a})^6; (2\sqrt[5]{x^4} - \frac{1}{2}y^3)^4; \left(\frac{2x^2}{y} - \sqrt[3]{y^2}\right)^6.$$

$$18. \left(\sqrt{ab} - \frac{c}{2\sqrt{b}}\right)^5; \left(\frac{a^2}{2c} - \frac{\sqrt{c}}{3}\right)^3; \left(a^2b - \frac{\sqrt{b}}{2a}\right)^4.$$

$$19. \left(\frac{a}{b}\sqrt{\frac{c}{d}} - \sqrt{\frac{d^2}{c^2}}\right)^3; (a^{\frac{n}{2}} - a^{-\frac{n}{2}})^4; \left(\frac{2a}{b^2} - \frac{1}{3}b\sqrt{a}\right)^4.$$

$$20. \left(\sqrt{\frac{a}{bc}} - \frac{\sqrt{c}}{3ab}\right)^3; \left(\frac{\sqrt{a}}{2\sqrt[3]{b^2}} - 3\sqrt{b}\right)^3; \left(\frac{a\sqrt{a}}{\sqrt[6]{b^5}} - \frac{\sqrt[4]{b}}{2a}\right)^3.$$

Find the square root of :

$$21. x^{4m} + 6x^{3m}y^n + 11x^{2m}y^{2n} + 6x^my^{3n} + y^{4n}.$$

$$22. 1 + 4x^{-\frac{1}{2}} - 2x^{-\frac{3}{2}} - 4x^{-1} + 25x^{-\frac{3}{2}} - 24x^{-\frac{5}{2}} + 16x^{-2}.$$

277. If we wish to find the approximate value of $\frac{3}{\sqrt{2}}$, it will be less labor to multiply first both numerator and denominator by a factor that will render the denominator *rational*; in this case by $\sqrt{2}$. Thus,

$$\frac{3}{\sqrt{2}} = \frac{3\sqrt{2}}{\sqrt{2} \times \sqrt{2}} = \frac{3\sqrt{2}}{2}.$$

278. It is easy to rationalize the denominator of a fraction when that denominator is a *binomial* involving only quadratic surds. The factor required will consist of the same terms as the given denominator, but with a different

sign between them. Thus, $\frac{7-3\sqrt{5}}{6+2\sqrt{5}}$ will have its denominator rationalized by multiplying both terms of the fraction by $6-2\sqrt{5}$. For,

$$\begin{aligned}\frac{7-3\sqrt{5}}{6+2\sqrt{5}} &= \frac{(7-3\sqrt{5})(6-2\sqrt{5})}{(6+2\sqrt{5})(6-2\sqrt{5})} \\ &= \frac{72-32\sqrt{5}}{16} = \frac{9}{2} - 2\sqrt{5}.\end{aligned}$$

279. By two operations the denominator of a fraction may be rationalized when that denominator consists of *three* quadratic surds.

Thus, if the denominator is $\sqrt{6} + \sqrt{3} - \sqrt{2}$, both terms of the fraction may be multiplied by $\sqrt{6} - \sqrt{3} + \sqrt{2}$. The resulting denominator will be $6 - 5 + 2\sqrt{6} = 1 + 2\sqrt{6}$; and if both terms of the resulting fraction be multiplied by $1 - 2\sqrt{6}$, the denominator will become $1 - 24 = -23$.

EXERCISE 104.

Find equivalent fractions with rational denominators, for the following:

$$1. \frac{3}{\sqrt{7} + \sqrt{5}}; \frac{7}{2\sqrt{5} - \sqrt{6}}; \frac{4 - \sqrt{2}}{1 + \sqrt{2}}; \frac{6}{5 - 2\sqrt{6}}.$$

$$2. \frac{a}{\sqrt{b} - \sqrt{c}}; \frac{a + b}{a - \sqrt{b}}; \frac{2x - \sqrt{xy}}{\sqrt{xy} - 2y}.$$

Find the approximate values of:

$$3. \frac{2}{\sqrt{3}}; \frac{1}{\sqrt{5} - \sqrt{2}}; \frac{7\sqrt{5}}{\sqrt{7} + \sqrt{3}}; \frac{7 + 2\sqrt{10}}{7 - 2\sqrt{10}}.$$

IMAGINARY EXPRESSIONS.

280. All imaginary square roots may be reduced to one form.

$$\sqrt{-x^2} = \sqrt{x^2 \times (-1)} = x\sqrt{-1}.$$

$$\sqrt{-a} = \sqrt{a \times (-1)} = a^{\frac{1}{2}}\sqrt{-1}.$$

281. $\sqrt{-1}$ means an expression which, when multiplied by itself, produces -1 . Therefore,

$$(\sqrt{-1})^2 = -1;$$

$$(\sqrt{-1})^3 = (\sqrt{-1})^2 \times \sqrt{-1} = -1 \sqrt{-1} = -\sqrt{-1};$$

$$(\sqrt{-1})^4 = (\sqrt{-1})^2 \times (\sqrt{-1})^2 = (-1) \times (-1) = 1;$$

and so on. So that the successive powers of $\sqrt{-1}$ form the repeating series, $+\sqrt{-1}, -1, -\sqrt{-1}, +1$.

(1) Multiply $1 + \sqrt{-4}$ by $1 - \sqrt{-4}$.

$$1 + \sqrt{-4} = 1 + 2\sqrt{-1};$$

$$1 - \sqrt{-4} = 1 - 2\sqrt{-1}.$$

$$(1 + 2\sqrt{-1})(1 - 2\sqrt{-1}) = 1 - 4(-1) = 5.$$

(2) Divide $\sqrt{-ab}$ by $\sqrt{-b}$.

$$\sqrt{-ab} = a^{\frac{1}{2}}b^{\frac{1}{2}}\sqrt{-1},$$

$$\text{and} \quad \sqrt{-b} = b^{\frac{1}{2}}\sqrt{-1}.$$

$$\frac{\sqrt{-ab}}{\sqrt{-b}} = \frac{a^{\frac{1}{2}}b^{\frac{1}{2}}\sqrt{-1}}{b^{\frac{1}{2}}\sqrt{-1}} = \sqrt{a}.$$

EXERCISE 105.

Multiply:

1. $4 + \sqrt{-3}$ by $4 - \sqrt{-3}$; $\sqrt{3} - 2\sqrt{-2}$ by $\sqrt{3} + 2\sqrt{-2}$.

2. $\sqrt{54}$ by $\sqrt{-2}$; $a\sqrt{-b}$ by $x\sqrt{-y}$.

3. $\sqrt{-a} + \sqrt{-b}$ by $\sqrt{-a} - \sqrt{-b}$; $a\sqrt{-a^2b^4}$ by $\sqrt{-a^4b^5}$.

4. $\sqrt{-10}$ by $\sqrt{-2}$; $2\sqrt{3} - 6\sqrt{-5}$ by $4\sqrt{3} - \sqrt{-5}$.

Divide:

5. $x\sqrt{-1}$ by $y\sqrt{-1}$; 1 by $\sqrt{-1}$; a by $a^{\frac{1}{2}}\sqrt{-1}$.

6. $\sqrt{-12}$ by $\sqrt{-3}$; $\sqrt{15}$ by $\sqrt{-5}$; $\sqrt{-5}$ by $\sqrt{-20}$.

SQUARE ROOT OF A BINOMIAL SURD.

282. *The product or quotient of two dissimilar quadratic surds will be a quadratic surd. Thus,*

$$\sqrt{ab} \times \sqrt{abc} = ab\sqrt{c};$$

$$\sqrt{abc} \div \sqrt{ab} = \sqrt{c}.$$

For every quadratic surd, when simplified, will have under the radical sign one or more factors raised only to the first power; and two surds which are *dissimilar* cannot have all these factors alike.

Hence, their product or quotient will have *at least one factor* raised only to the *first power*, and will therefore be a surd.

283. *The sum or difference of two dissimilar quadratic surds cannot be a rational number, nor can it be expressed as a single surd.*

For if $\sqrt{a} \pm \sqrt{b}$ could equal a rational number c , we should have, by squaring,

$$a \pm 2\sqrt{ab} + b = c^2;$$

that is,

$$\pm 2\sqrt{ab} = c^2 - a - b.$$

Now, as the right side of this equation is rational, the left side would be rational; but, by § 282, \sqrt{ab} cannot be rational. Therefore, $\sqrt{a} \pm \sqrt{b}$ cannot be rational.

In like manner, it may be shown that $\sqrt{a} \pm \sqrt{b}$ cannot be expressed as a single surd \sqrt{c} .

284. *A quadratic surd cannot equal the sum of a rational number and a surd.*

For, if \sqrt{a} could equal $c + \sqrt{b}$, we should have, by squaring,

$$a = c^2 + 2c\sqrt{b} + b,$$

and, by transposing, $2c\sqrt{b} = a - b - c^2$.

That is, a surd equal to a rational number, which is impossible.

285. *If $a + \sqrt{b} = x + \sqrt{y}$, then a will equal x and b will equal y .*

For, by transposing, $\sqrt{b} - \sqrt{y} = x - a$; and if b were not equal to y , the difference of two unequal surds would be rational, which by § 283 is impossible.

$$\therefore b = y, \text{ and } a = x.$$

In like manner, if $a - \sqrt{b} = x - \sqrt{y}$, a will equal x and b will equal y .

286. *To extract the square root of a binomial surd $a + \sqrt{b}$.*

$$\text{Let } \sqrt{a + \sqrt{b}} = \sqrt{x} + \sqrt{y}.$$

$$\text{Squaring, } a + \sqrt{b} = x + 2\sqrt{xy} + y.$$

$$\therefore x + y = a, \text{ and } 2\sqrt{xy} = \sqrt{b}. \quad \S 285.$$

From these two equations the values of x and y may be found.

This method may be shortened by observing that, since $\sqrt{b} = 2\sqrt{xy}$,

$$a - \sqrt{b} = x - 2\sqrt{xy} + y.$$

$$\text{By taking the root, } \sqrt{a - \sqrt{b}} = \sqrt{x} - \sqrt{y}.$$

$$\therefore (\sqrt{a + \sqrt{b}})(\sqrt{a - \sqrt{b}}) = (\sqrt{x} + \sqrt{y})(\sqrt{x} - \sqrt{y}).$$

$$\therefore \sqrt{a^2 - b} = x - y.$$

And, as

$$a = x + y,$$

the values of x and y may be found by addition and subtraction.

Ex. Extract the square root of $7 + 4\sqrt{3}$.

$$\text{Let} \quad \sqrt{x} + \sqrt{y} = \sqrt{7 + 4\sqrt{3}}.$$

$$\text{Then} \quad \sqrt{x} - \sqrt{y} = \sqrt{7 - 4\sqrt{3}}$$

$$\text{By multiplying,} \quad x - y = \sqrt{49 - 48}.$$

$$\therefore x - y = 1.$$

$$\text{But} \quad x + y = 7.$$

$$\therefore x = 4, \text{ and } y = 3.$$

$$\therefore \sqrt{x} + \sqrt{y} = 2 + \sqrt{3}.$$

$$\therefore \sqrt{7 + 4\sqrt{3}} = 2 + \sqrt{3}.$$

EXERCISE 106.

Extract the square roots of:

- | | | |
|----------------------------------|---------------------------------------|----------------------------------|
| 1. $14 + 6\sqrt{5}$. | 6. $20 - 8\sqrt{6}$. | 11. $14 - 4\sqrt{6}$. |
| 2. $17 + 4\sqrt{15}$. | 7. $9 - 6\sqrt{2}$. | 12. $38 - 12\sqrt{10}$. |
| 3. $10 + 2\sqrt{21}$. | 8. $94 - 42\sqrt{5}$. | 13. $103 - 12\sqrt{11}$. |
| 4. $16 + 2\sqrt{55}$. | 9. $13 - 2\sqrt{30}$. | 14. $57 - 12\sqrt{15}$. |
| 5. $9 - 2\sqrt{14}$. | 10. $11 - 6\sqrt{2}$. | 15. $3\frac{1}{2} - \sqrt{10}$. |
| 16. $2a + 2\sqrt{a^2 - b^2}$. | 18. $87 - 12\sqrt{42}$. | |
| 17. $a^2 - 2b\sqrt{a^2 - b^2}$. | 19. $(a + b)^2 - 4(a - b)\sqrt{ab}$. | |

287. A root may often be obtained by inspection. For this purpose, write the given expression in the form $a + 2\sqrt{b}$, and determine what two numbers have their sum equal to a , and their product equal to b .

(1) Find by inspection the square root of $18 + 2\sqrt{77}$.

It is required to find two numbers whose sum is 18 and whose product is 77; and these are evidently 11 and 7.

$$\begin{aligned} \text{Then} \quad 18 + 2\sqrt{77} &= 11 + 7 + 2\sqrt{11 \cdot 7}, \\ &= (\sqrt{11} + \sqrt{7})^2. \end{aligned}$$

$$\text{That is,} \quad \sqrt{11} + \sqrt{7} = \text{square root of } 18 + 2\sqrt{77}.$$

(2) Find by inspection the square root of $75 - 12\sqrt{21}$.

It is necessary that the coefficient of the surd be 2; therefore, $75 - 12\sqrt{21}$ must be put in the form of

$$75 - 2\sqrt{6^2 \times 21} = 75 - 2\sqrt{756}.$$

The two numbers whose sum is 75 and whose product is 756 are 63 and 12.

$$\begin{aligned}\text{Then } 75 - 2\sqrt{756} &= 63 + 12 - 2\sqrt{63 \times 12}, \\ &= (\sqrt{63} - \sqrt{12})^2.\end{aligned}$$

That is, $\sqrt{63} - \sqrt{12}$ = the square root of $75 - 12\sqrt{21}$;

or, $3\sqrt{7} - 2\sqrt{3}$ = the square root of $75 - 12\sqrt{21}$.

EQUATIONS CONTAINING RADICALS.

288. An equation containing a *single* radical may be solved by arranging the terms so as to have the radical alone on one side, and then raising both sides to a power corresponding to the order of the radical.

Ex. $\sqrt{x^2 - 9} + x = 9$.

$$\begin{aligned}\sqrt{x^2 - 9} &= 9 - x. \\ \text{By squaring, } x^2 - 9 &= 81 - 18x + x^2, \\ 18x &= 90. \\ \therefore x &= 5\end{aligned}$$

289. If *two* radicals are involved, two steps may be necessary.

Ex. $\sqrt{x+15} + \sqrt{x} = 15$.

$$\begin{aligned}\sqrt{x+15} + \sqrt{x} &= 15. \\ \text{By squaring, } x + 15 + 2\sqrt{x^2 + 15x} + x &= 225. \\ \text{By transposing, } 2\sqrt{x^2 + 15x} &= 210 - 2x. \\ \text{By dividing by 2, } \sqrt{x^2 + 15x} &= 105 - x. \\ \text{By squaring, } x^2 + 15x &= 11025 - 210x + x^2. \\ 225x &= 11025. \\ \therefore x &= 49.\end{aligned}$$

Some of the following radical equations will reduce to simple and others to quadratic equations.

Solve:

EXERCISE 107.

1. $\sqrt{x-5}=2$.
2. $2\sqrt{3x+4}-x=4$.
3. $3-\sqrt{x^2-1}=2x$.
4. $\sqrt{3x-2}=2(x-4)$.
5. $4x-12\sqrt{x}=16$.
6. $\sqrt{x+4}+\sqrt{2x-1}=6$.
7. $\sqrt{13x-1}-\sqrt{2x-1}=5$.
8. $\sqrt{4+x}+\sqrt{x}=3$.
9. $\sqrt{25+x}+\sqrt{25-x}=8$.
10. $x^2=21+\sqrt{x^2-9}$.
11. $2x-\sqrt[3]{8x^3+26}+2=0$.
12. $\sqrt{x+1}+\sqrt{x+16}=\sqrt{x+25}$.
13. $\sqrt{2x+1}-\sqrt{x+4}=\frac{1}{8}\sqrt{x-3}$.
14. $\sqrt{x+3}+\sqrt{x+8}=5\sqrt{x}$.
15. $\sqrt{3+x}+\sqrt{x}=\frac{6}{\sqrt{3+x}}$.
16. $\sqrt{x^2-1}+6=\frac{16}{\sqrt{x^2-1}}$.
17. $\frac{1}{\sqrt{x+1}}+\frac{1}{\sqrt{x-1}}=\frac{1}{\sqrt{x^2-1}}$.
18. $\frac{\sqrt{x+2a}-\sqrt{x-2a}}{\sqrt{x-2a}+\sqrt{x+2a}}=\frac{x}{2a}$.
19. $\frac{3x+\sqrt{4x-x^2}}{3x-\sqrt{4x-x^2}}=2$.
20. $\frac{\sqrt{7x^2+4}+2\sqrt{3x-1}}{\sqrt{7x^2+4}-2\sqrt{3x-1}}=7$.
21. $\sqrt{(x-a)^2+2ab+b^2}=x-a+b$.
22. $\sqrt{(x+a)^2+2ab+b^2}=b-a-x$.
23. $\sqrt{\frac{x}{4}+3}+\sqrt{\frac{x}{4}-3}=\sqrt{\frac{2x}{3}}$.

$$24. \quad 4x^{\frac{1}{2}} - 3(x^{\frac{1}{2}} + 1)(x^{\frac{1}{2}} - 2) = x^{\frac{1}{2}}(10 - 3x^{\frac{1}{2}}).$$

$$25. \quad (x^{\frac{1}{2}} - 2)(x^{\frac{1}{2}} - 4) = x^{\frac{1}{2}}(x^{\frac{1}{2}} - 1)^2 - 12.$$

$$26. \quad x^3 - 4x^{\frac{1}{2}} = 96.$$

$$28. \quad x^{\frac{1}{2}} + 2a^2x^{-\frac{1}{2}} = 3a.$$

$$27. \quad x + x^{-1} = 2.9.$$

$$29. \quad 81\sqrt[3]{x} + \frac{81}{\sqrt[3]{x}} = 52x.$$

290. Equations may be solved with respect to an *expression* in the same manner as with respect to a letter.

$$(1) \quad \text{Solve } (x^2 - x)^2 - 8(x^2 - x) + 12 = 0.$$

Consider $(x^2 - x)$ as the unknown number.

Then $(x^2 - x)^2 - 8(x^2 - x) = -12.$

Complete the square, $(x^2 - x)^2 - () + 16 = 4.$

Extract the root, $(x^2 - x) - 4 = \pm 2.$

$$x^2 - x = 6 \text{ or } 2.$$

Complete the square, $4x^2 - () + 1 = 25 \text{ or } 9.$

Extract the root, $2x - 1 = \pm 5 \text{ or } \pm 3.$

$$2x = 6, -4, 4, -2.$$

$$\therefore x = 3, -2, 2, -1.$$

$$(2) \quad \text{Solve } 5x - 7x^2 - 8\sqrt{7x^2 - 5x + 1} = 8.$$

Change the signs and annex $+1$ to both sides.

$$7x^2 - 5x + 1 + 8\sqrt{7x^2 - 5x + 1} = -7.$$

Solve with respect to $\sqrt{7x^2 - 5x + 1}.$

$$(7x^2 - 5x + 1) + 8(7x^2 - 5x + 1)^{\frac{1}{2}} + 16 = 9.$$

$$(7x^2 - 5x + 1)^{\frac{1}{2}} + 4 = \pm 3.$$

$$(7x^2 - 5x + 1)^{\frac{1}{2}} = -1 \text{ or } -7.$$

Square, $7x^2 - 5x + 1 = 1 \text{ or } 49.$

Transpose, $7x^2 - 5x = 0 \text{ or } 48.$

From $7x^2 - 5x = 0$, $x = 0 \text{ or } \frac{5}{7};$

From $7x^2 - 5x = 48$, $x = 3 \text{ or } -2\frac{2}{7}.$

NOTE. In verifying the values of x in the original equation, it is seen that the value of $\sqrt{7x^2 - 5x + 1}$ is negative. Thus, by putting 0 for x the equation becomes $0 - 8\sqrt{1} = 8$; and by taking -1 for $\sqrt{1}$ we have $(-8)(-1) = 8$; that is, $8 = 8.$

(3) Solve $x^2 + x + 1 + \frac{1}{x} + \frac{1}{x^2} = 1$.

Arrange as follows: $\left(x^2 + \frac{1}{x^2}\right) + \left(x + \frac{1}{x}\right) = 0$.

By adding 2 to $\left(x + \frac{1}{x}\right)$,

there is obtained $x^2 + 2 + \frac{1}{x^2} = \left(x + \frac{1}{x}\right)^2$.

$$\therefore \left(x + \frac{1}{x}\right)^2 + \left(x + \frac{1}{x}\right) = 2.$$

Multiply by 4 and complete the square,

$$4\left(x + \frac{1}{x}\right)^2 + () + 1 = 9.$$

Extract the root, $2\left(x + \frac{1}{x}\right) + 1 = \pm 3$.

$$2\left(x + \frac{1}{x}\right) = 2 \text{ or } -4.$$

$$x + \frac{1}{x} = 1 \text{ or } -2.$$

Multiply by x , $x^2 - x = -1$, and $x^2 + 2x = -1$.

$$\therefore 4x^2 - () + 1 = -3, \quad \therefore x^2 + 2x + 1 = 0.$$

$$2x - 1 = \pm \sqrt{-3}, \quad x + 1 = 0.$$

$$\therefore x = \frac{1}{2}(1 \pm \sqrt{-3}). \quad \therefore x = -1.$$

291. An equation like that of (3) which will remain unaltered when $\frac{1}{x}$ is substituted for x , is called a **reciprocal equation**.

It will be found that every reciprocal equation of *odd* degree will be divisible by $x - 1$ or $x + 1$ according as the last term is negative or positive; and every reciprocal equation of *even* degree *with its last term negative* will be divisible by $x^2 - 1$. In every case the equation resulting from the division will be reciprocal.

(4) Solve $x^5 + 2x^4 - 3x^3 - 3x^2 + 2x + 1 = 0$.

This is a reciprocal equation, for, if x^{-1} is put for x , the equation becomes $x^{-5} + 2x^{-4} - 3x^{-3} - 3x^{-2} + 2x^{-1} + 1 = 0$, which multiplied by x^5 gives $1 + 2x - 3x^2 - 3x^3 + 2x^4 + x^5 = 0$, the same as the original equation.

The equation may be written $(x^5 + 1) + 2x(x^3 + 1) - 3x^2(x + 1) = 0$, which is obviously divisible by $x + 1$. The result from dividing by $x + 1$ is $x^4 + x^3 - 4x^2 + x + 1 = 0$, or $(x^4 + 1) + x(x^2 + 1) = 4x^2$. By adding $2x^2$ to $(x^4 + 1)$ it becomes $(x^4 + 2x^2 + 1) = (x^2 + 1)^2$.

Then $(x^2 + 1)^2 + x(x^2 + 1) = 6x^2$.

Multiply by 4 and complete the square,

$$4(x^2 + 1)^2 + () + x^2 = 25x^2.$$

Extract the root, $2(x^2 + 1) + x = \pm 5x$

Hence, $2x^2 + 2 = 4x$ or $-6x$.

By simplifying, $x^2 - 2x = -1$; and $x^2 + 3x = -1$,

whence, $x = 1$ and 1 ; whence, $x = \frac{1}{2}(-3 \pm \sqrt{5})$.

Therefore, including the root -1 obtained from the factor $x + 1$, the five roots are $-1, 1, 1, \frac{1}{2}(-3 \pm \sqrt{5})$.

By this process a reciprocal *cubic* equation may be reduced to a *quadratic*, and one of the *fifth* or *sixth* degree to a *biquadratic*, the solution of which may be easily effected.

Solve : EXERCISE 108.

1. $x^2 - 3x - 6\sqrt{x^2 - 3x - 3} + 2 = 0$.

2. $x^2 + 3x - \frac{3}{x} + \frac{1}{x^2} = \frac{7}{36}$.

3. $(2x^2 - 3x)^2 - 2(2x^2 - 3x) = 15$.

4. $(ax - b)^2 + 4a(ax - b) = \frac{9a^2}{4}$.

5. $3(2x^2 - x) - (2x^2 - x)^{\frac{1}{2}} = 2$.

6. $15x - 3x^2 + 4(x^2 - 5x + 5)^{\frac{1}{2}} = 16$.

7. $x^3 + x^{-2} + x + x^{-1} = 4$. 9. $x^2 + x + \frac{1}{8}(x^2 + x)^{\frac{1}{2}} = \frac{7}{8}$.

8. $x^3 + \sqrt{x^2 - 7} = 19$. 10. $(x + 1)^{\frac{1}{2}} + (x - 1)^{\frac{1}{2}} = 5$.

$$11. x-1=2+2x^{-1}. \quad 12. \sqrt{3x+5}-\sqrt{3x-5}=4.$$

$$13. (x^4+1)-x(x^2+1)=-2x^2.$$

$$14. 2x^2-2\sqrt{2x^2-5x}=5(x+3).$$

$$15. x+2-4x\sqrt{x+2}=12x^2.$$

$$16. \sqrt{2x+a}+\sqrt{2x-a}=b.$$

$$17. \sqrt{9x^2+21x+1}-\sqrt{9x^2+6x+1}=3x.$$

$$18. x^{\frac{1}{2}}-4x^{\frac{1}{2}}+x^{-\frac{1}{2}}+4x^{-\frac{1}{2}}=-\frac{7}{4}.$$

$$19. \left. \begin{aligned} (2x+3y)^2-2(2x+3y) &= 8 \\ x^2-y^2 &= 21 \end{aligned} \right\}$$

$$20. \left. \begin{aligned} x+y+\sqrt{x+y} &= a \\ x-y+\sqrt{x-y} &= b \end{aligned} \right\} \quad 22. \left. \begin{aligned} x^2+y^2+x+y &= 48 \\ xy &= 12 \end{aligned} \right\}$$

$$21. \left. \begin{aligned} x^4-x^2y^2+y^4 &= 13 \\ x^2-xy+y^2 &= 3 \end{aligned} \right\} \quad 23. \left. \begin{aligned} x^2+xy+y^2 &= a^2 \\ x+\sqrt{xy+y} &= b \end{aligned} \right\}$$

$$24. \left. \begin{aligned} (x-y)^2-3(x-y) &= 10 \\ x^2y^2-3xy &= 54 \end{aligned} \right\}$$

$$25. \left. \begin{aligned} \sqrt{x}-\sqrt{y} &= x^{\frac{1}{2}}(x^{\frac{1}{2}}+y^{\frac{1}{2}}) \\ (x+y)^2 &= 2(x-y)^2 \end{aligned} \right\}$$

$$26. \left. \begin{aligned} \left(\frac{3x}{x+y}\right)^{\frac{1}{2}}+\left(\frac{x+y}{3x}\right)^{\frac{1}{2}} &= 2 \\ xy-(x+y) &= 54 \end{aligned} \right\}$$

$$27. \left. \begin{aligned} x+y+\sqrt{xy} &= 28 \\ x^2+y^2+xy &= 336 \end{aligned} \right\}$$

$$28. \frac{x^3}{a}-3ax=\sqrt{4x^3+9ax^2}+\frac{27a^2}{4}.$$

$$29. (x+1+x^{-1})(x-1+x^{-1})=5\frac{1}{4}.$$

$$30. 2(x^{\frac{1}{2}}-1)^{-1}-2(x^{\frac{1}{2}}-4)^{-1}=3(x^{\frac{1}{2}}-2)^{-1}.$$

CHAPTER XIX.

LOGARITHMS.

292. In the common system of notation the expression of numbers is founded on their relation to *ten*.

Thus, 3854 indicates that this number contains 10^3 three times, 10^2 eight times, 10 five times, and four units.

293. In this system a number is represented by a series of *different* powers of ten, the exponent of each power being *integral*. But, by employing *fractional* exponents, any number may be represented (approximately) as a *single* power of 10.

294. When numbers are referred in this way to 10, the **exponents of the powers** corresponding to them are called their **logarithms** to the base 10.

For brevity the word "logarithm" is written log.

From § 255 it appears that:

$$\begin{array}{ll} 10^0 = 1, & 10^{-1} (= \frac{1}{10}) = 0.1, \\ 10^1 = 10, & 10^{-2} (= \frac{1}{100}) = 0.01, \\ 10^2 = 100, & 10^{-3} (= \frac{1}{1000}) = 0.001, \end{array}$$

and so on. Hence,

$$\begin{array}{ll} \log 1 = 0, & \log 0.1 = -1, \\ \log 10 = 1, & \log 0.01 = -2, \\ \log 100 = 2, & \log 0.001 = -3, \end{array}$$

and so on.

It is evident that the logarithms of all numbers between

1 and 10 will be $0 +$ a fraction,
 10 and 100 will be $1 +$ a fraction,
 100 and 1000 will be $2 +$ a fraction,
 1 and 0.1 will be $-1 +$ a fraction,
 0.1 and 0.01 will be $-2 +$ a fraction,
 0.01 and 0.001 will be $-3 +$ a fraction.

295. The fractional part of a logarithm cannot be expressed *exactly* either by common or by decimal fractions: but decimals may be obtained for these fractional parts, true to as many places as may be desired.

If, for instance, the logarithm of 2 be required; $\log 2$ may be supposed to be $\frac{1}{10}$.

Then $10^{\frac{1}{10}} = 2$; or, by raising both sides to the *third* power, $10 = 8$, a result which shows that $\frac{1}{10}$ is too large.

Suppose, then, $\log 2 = \frac{3}{10}$. Then $10^{\frac{3}{10}} = 2$, or by raising both sides to the *tenth* power, $10^3 = 2^{10}$. That is, $1000 = 1024$, a result which shows that $\frac{3}{10}$ is too small.

Since $\frac{1}{10}$ is too large and $\frac{3}{10}$ too small, $\log 2$ lies between $\frac{1}{10}$ and $\frac{3}{10}$; that is, between 0.33333 and 0.30000.

In supposing $\log 2$ to be $\frac{1}{10}$, the error of the result is $\frac{10-8}{10} = \frac{2}{10} = 0.2$. In supposing $\log 2$ to be $\frac{3}{10}$, the error of the result is $\frac{1000-1024}{1000} = -0.024$; $\log 2$, therefore, is nearer to $\frac{3}{10}$ than to $\frac{1}{10}$.

The difference between the errors is $0.2 - (-0.024) = 0.224$, and the difference between the supposed logarithms is $0.33333 - 0.3 = 0.03333$.

The last error, therefore, in the supposed logarithm may be considered to be approximately $\frac{2}{224}$ of $0.03333 = 0.0035$ nearly, and this added to 0.3000 gives 0.3035, a result a little too large.

By shorter methods of higher mathematics, the logarithm of 2 is known to be 0.3010300, true to the seventh place.

296. The logarithm of a number consists of two parts, an integral part and a fractional part.

Thus, $\log 2 = 0.30103$, in which the integral part is 0, and the fractional part is .30103; $\log 20 = 1.30103$, in which the integral part is 1, and the fractional part is .30103.

297. The integral part of a logarithm is called the **characteristic**; and the fractional part is called the **mantissa**.

298. The mantissa is always made *positive*. Hence, in the case of numbers less than 1 whose logarithms are *negative*, the logarithm is made to consist of a *negative* characteristic and a *positive* mantissa.

299. When a logarithm consists of a *negative* characteristic and a *positive* mantissa, it is usual to write the minus sign *over* the characteristic, or else to add 10 to the characteristic and to indicate the subtraction of 10 from the resulting logarithm.

Thus, $\log .02 = \bar{1}.30103$, and this may be written $9.30103 - 10$.

300. *The characteristic of a logarithm of an integral number, or of a mixed number, is one less than the number of integral digits.*

Thus, from § 294, $\log 1 = 0$, $\log 10 = 1$, $\log 100 = 2$. Hence, the logarithms of all numbers from 1 to 10 (that is, of all numbers consisting of *one* integral digit), will have 0 for a characteristic; and the logarithms of all numbers from 10 to 100 (that is, of all numbers consisting of *two* integral digits), will have 1 for a characteristic; and so on, the characteristic increasing by 1 for each increase in the number of digits, and therefore always being 1 less than that number.

301. *The characteristic of a logarithm of a decimal fraction is negative, and is equal to the number of the place occupied by the first significant figure of the decimal.*

Thus, from § 294, $\log 0.1 = -1$, $\log 0.01 = -2$, $\log 0.001 = -3$. Hence, the logarithms of all numbers from 0.1 to 1 will have -1 for a characteristic (the mantissa being *plus*); the logarithms of all numbers from 0.01 to 0.1 will have -2 for a characteristic; the logarithms of all numbers from 0.001 to 0.01 will have -3 for a characteristic; and so on, the characteristic always being *negative and equal to the number of the place occupied by the first significant figure of the decimal*.

302. *The mantissa of a logarithm of any integral number or decimal fraction depends only upon the digits of the number, and is unchanged so long as the sequence of the digits remains the same.*

For, changing the position of the decimal point in a number is equivalent to multiplying or dividing the number by a power of 10. Its logarithm, therefore, will be increased or diminished by the *exponent* of that power of 10; and, since this exponent is *integral*, the *mantissa* of the logarithm will be unaffected.

Thus,	if	$27196 = 10^{4.4345},$
then		$2719.6 = 10^{3.4345},$
		$27.196 = 10^{1.4345},$
		$2.7196 = 10^{0.4345},$
		$0.27196 = 10^{9.4345} - 10,$
		$0.0027196 = 10^{7.4345} - 10.$

303. The advantage of using the number 10 as the base of a system of logarithms consists in the fact that the **mantissa** depends only on the **sequence of digits**, and the **characteristic** on the **position of the decimal point**.

304. As logarithms are simply exponents (§ 294), therefore,

The logarithm of a product is the sum of the logarithms of the factors.

Thus, $\log 20 = \log (2 \times 10) = \log 2 + \log 10,$	
	$= 0.3010 + 1.0000 = 1.3010;$
$\log 2000 = \log (2 \times 1000) = \log 2 + \log 1000.$	
	$= 0.3010 + 3.0000 = 3.3010;$
$\log 0.2 = \log (2 \times 0.1) = \log 2 + \log 0.1,$	
	$= 0.3010 + 9.0000 - 10 = 9.3010 - 10,$
$\log 0.02 = \log (2 \times 0.01) = \log 2 + \log 0.01,$	
	$= 0.3010 + 8.0000 - 10 = 8.3010 - 10.$

EXERCISE 109.

Given: $\log 2 = 0.3010$; $\log 3 = 0.4771$; $\log 5 = 0.6990$;
 $\log 7 = 0.8451$.

Find the logarithms of the following numbers by resolv-

ing the numbers into factors, and taking the sum of the logarithms of the factors :

1. log 6.	9. log 25.	17. log 0.021.	25. log 2.1.
2. log 15.	10. log 30.	18. log 0.35.	26. log 16.
3. log 21.	11. log 42.	19. log 0.0035.	27. log 0.056.
4. log 14.	12. log 420.	20. log 0.004.	28. log 0.63.
5. log 35.	13. log 12.	21. log 0.05.	29. log 1.75.
6. log 9.	14. log 60.	22. log 12.5.	30. log 105.
7. log 8.	15. log 75.	23. log 1.25.	31. log 0.0105.
8. log 49.	16. log 7.5.	24. log 37.5.	32. log 1.05.

305. As logarithms are simply exponents (§ 294), therefore,

The logarithm of a power of a number is equal to the logarithm of the number multiplied by the exponent of the power.

$$\begin{aligned}\text{Thus, } \log 5^7 &= 7 \times \log 5 = 7 \times 0.6990 = 4.8930. \\ \log 3^{11} &= 11 \times \log 3 = 11 \times 0.4771 = 5.2481.\end{aligned}$$

306. As logarithms are simply exponents (§ 294), therefore, when roots are expressed by fractional indices,

The logarithm of a root of a number is equal to the logarithm of the number multiplied by the index of the root.

$$\begin{aligned}\text{Thus, } \log 2^{\frac{1}{4}} &= \frac{1}{4} \text{ of } \log 2 = \frac{1}{4} \times 0.3010 = 0.0753. \\ \log 0.002^{\frac{1}{3}} &= \frac{1}{3} \text{ of } (7.3010 - 10).\end{aligned}$$

The expression $\frac{1}{3}$ of $(7.3010 - 10)$ may be put in the form of $\frac{1}{3}$ of $(27.3010 - 30)$ which $= 9.1003 - 10$; for, since $20 - 20 = 0$, the addition of 20 to the 7, and of -20 to the -10 , produces no change in the value of the logarithm.

307. *In simplifying the logarithm of a root the equal positive and negative numbers to be added to the logarithm must be such that the resulting negative number, when divided by the index of the root, shall give a quotient of -10 .*

EXERCISE 110.

Given: $\log 2 = 0.3010$; $\log 3 = 0.4771$; $\log 5 = 0.6990$;
 $\log 7 = 0.8451$.

Find logarithms of the following:

1. 2^3 .	6. 5^5 .	11. $5\frac{1}{2}$.	16. $7\frac{1}{2}$.	21. $5\frac{1}{2}$.
2. 5^3 .	7. $2\frac{1}{2}$.	12. $7\frac{1}{2}$.	17. $5\frac{1}{2}$.	22. $2\frac{1}{2}$.
3. 7^4 .	8. $5\frac{1}{2}$.	13. $2\frac{1}{2}$.	18. $3\frac{1}{2}$.	23. $5\frac{1}{2}$.
4. 3^8 .	9. $3\frac{1}{2}$.	14. $5\frac{1}{2}$.	19. $7\frac{1}{2}$.	24. $7\frac{1}{2}$.
5. 7^3 .	10. $7\frac{1}{2}$.	15. $3\frac{1}{2}$.	20. $3\frac{1}{2}$.	25. $21\frac{1}{2}$.

308. Since logarithms are simply exponents (§ 294), therefore,

The logarithm of a quotient is the logarithm of the dividend minus the logarithm of the divisor.

$$\begin{aligned}\text{Thus, } \log \frac{3}{2} &= \log 3 - \log 2 = 0.4771 - 0.3010 = 0.1761. \\ \log \frac{2}{3} &= \log 2 - \log 3 = 0.3010 - 0.4771 = -0.1761.\end{aligned}$$

To avoid the negative logarithm -0.1761 , we subtract the *entire* logarithm 0.1761 from 10, and then indicate the subtraction of 10 from the result.

$$\begin{aligned}\text{Thus,} \quad & -0.1761 = 9.8239 - 10. \\ \text{Hence,} \quad & \log \frac{2}{3} = 9.8239 - 10.\end{aligned}$$

309. The remainder obtained by subtracting the logarithm of a number from 10 is called the **cologarithm** of the number, or **arithmetical complement** of the logarithm of the number.

Cologarithm is usually denoted by *colog*, and is most easily found by *beginning with the characteristic of the logarithm and subtracting each figure from 9 down to the last significant figure, and subtracting that figure from 10.*

Thus, $\log 7 = 0.8451$; and $\text{colog } 7 = 9.1549$. Colog 7 is readily found by subtracting, mentally, 0 from 9, 8 from 9, 4 from 9, 5 from 9, 1 from 10, and writing the resulting figure at each step.

310. Since $\text{colog } 7 = 9.1549$,
and $\log \frac{1}{7} = \log 1 - \log 7 = 0 - 0.8451 = 9.1549 - 10$,
it is evident that,

If 10 be subtracted from the cologarithm of a number, the result is the logarithm of the reciprocal of that number.

311. Since $\log \frac{7}{5} = \log 7 - \log 5$,
 $\phantom{\log \frac{7}{5}} = 0.8451 - 0.6990 = 0.1461$,
and $\log 7 + \text{colog } 5 - 10 = 0.8451 + 9.3010 - 10$,
 $\phantom{\log 7 + \text{colog } 5 - 10} = 0.1461$,

it is evident that,

The addition of a cologarithm — 10 is equivalent to the subtraction of a logarithm.

The steps that lead to this result are :

$\frac{7}{5} = 7 \times \frac{1}{5}$,
therefore, $\log \frac{7}{5} = \log (7 \times \frac{1}{5}) = \log 7 + \log \frac{1}{5}$. § 304.
But $\log \frac{1}{5} = \text{colog } 5 - 10$. § 309.
Hence, $\log \frac{7}{5} = \log 7 + \text{colog } 5 - 10$.

Therefore,

312. *The logarithm of a quotient may be found by adding together the logarithm of the dividend and the cologarithm of the divisor, and subtracting 10 from the result.*

In finding a cologarithm when the *characteristic* of the logarithm is a *negative* number, it must be observed that the *subtraction* of a *negative* number is equivalent to the *addition* of an *equal positive* number.

Thus, $\log \frac{5}{0.002} = \log 5 + \text{colog } 0.002 - 10$,
 $\phantom{\log \frac{5}{0.002}} = 0.6990 + 12.6990 - 10$,
 $\phantom{\log \frac{5}{0.002}} = 3.3980$.

Here $\log 0.002 = \bar{3}.3010$, and in subtracting — 3 from 9 the result is the same as adding + 3 to 9.

Again, $\log \frac{2}{0.07} = \log 2 + \text{colog } 0.07 - 10$,
 $\phantom{\log \frac{2}{0.07}} = 0.3010 + 11.1549 - 10$,
 $\phantom{\log \frac{2}{0.07}} = 1.4559$.

$$\begin{aligned}
 \text{Also,} \quad \log \frac{0.07}{2^3} &= 8.8451 - 10 + 9.0970 - 10, \\
 &= 17.9421 - 20, \\
 &= 7.9421 - 10.
 \end{aligned}$$

$$\text{Here,} \quad \log 2^3 = 3 \log 2 = 3 \times 0.3010 = 0.9030.$$

$$\text{Hence,} \quad \text{colog } 2^3 = 10 - 0.9030 = 9.0970.$$

EXERCISE 111.

Given: $\log 2 = 0.3010$; $\log 3 = 0.4771$; $\log 5 = 0.6990$;
 $\log 7 = 0.8451$.

Find logarithms for the following quotients:

- | | | | | |
|------------------|---------------------|-----------------------|--------------------------|------------------------------|
| 1. $\frac{2}{5}$ | 7. $\frac{5}{3}$ | 13. $\frac{0.05}{3}$ | 19. $\frac{0.05}{0.003}$ | 25. $\frac{0.02^2}{3^3}$ |
| 2. $\frac{2}{7}$ | 8. $\frac{5}{2}$ | 14. $\frac{0.005}{2}$ | 20. $\frac{0.007}{0.02}$ | 26. $\frac{3^3}{0.02^2}$ |
| 3. $\frac{3}{5}$ | 9. $\frac{7}{3}$ | 15. $\frac{0.07}{5}$ | 21. $\frac{0.02}{0.007}$ | 27. $\frac{7^3}{0.02^2}$ |
| 4. $\frac{3}{7}$ | 10. $\frac{7}{2}$ | 16. $\frac{5}{0.07}$ | 22. $\frac{0.005}{0.07}$ | 28. $\frac{0.07^3}{0.003^3}$ |
| 5. $\frac{5}{7}$ | 11. $\frac{3}{2}$ | 17. $\frac{3}{0.007}$ | 23. $\frac{0.03}{7}$ | 29. $\frac{0.005^2}{7^3}$ |
| 6. $\frac{7}{5}$ | 12. $\frac{7}{0.5}$ | 18. $\frac{0.003}{7}$ | 24. $\frac{0.0007}{0.2}$ | 30. $\frac{7^3}{0.005^2}$ |

313. A table of *four-place* logarithms is here given, which contains logarithms of all numbers under 1000, *the decimal point and characteristic being omitted*. The logarithms of single digits 1, 8, etc., will be found at 10, 80, etc.

Tables containing logarithms of more places can be procured, but this table will serve for many practical uses, and will enable the student to use tables of six-place, seven-place, and ten-place logarithms in work that requires greater accuracy.

314. In working with a four-place table, the numbers corresponding to the logarithms, that is, the *antilogarithms*, as they are called, may be carried to *four significant digits*.

TO FIND THE LOGARITHM OF A NUMBER IN THIS TABLE.

315. Suppose it is required to find the logarithm of 65.7. In the column headed "N" look for the first two significant figures, and at the top of the table for the third significant figure. In the line with 65, and in the column headed 7, is seen 8176. To this number prefix the characteristic and insert the decimal point. Thus,

$$\log 65.7 = 1.8176.$$

Suppose it is required to find the logarithm of 20347. In the line with 20, and in the column headed 3, is seen 3075; also in the line with 20, and in the 4 column, is seen 3096, and the difference between these two is 21. The difference between 20300 and 20400 is 100, and the difference between 20300 and 20347 is 47. Hence, $\frac{47}{100}$ of 21 = 10, nearly, must be added to 3075. That is,

$$\log 20347 = 4.3085.$$

Suppose it is required to find the logarithm of 0.0005076. In the line with 50, and in the 7 column, is seen 7050; in the 8 column, 7059: the difference is 9. The difference between 5070 and 5080 is 10, and the difference between 5070 and 5076 is 6. Hence, $\frac{6}{10}$ of 9 = 5 must be added to 7050. That is, $\log 0.0005076 = 6.7055 - 10$.

TO FIND A NUMBER WHEN ITS LOGARITHM IS GIVEN.

316. Suppose it is required to find the number of which the logarithm is 1.9736.

Look for 9736 in the table. In the column headed "N," and in the line with 9736, is seen 94, and at the head of

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396

N	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996

the column in which 9736 stands is seen 1. Therefore, write 941, and insert the decimal point as the characteristic directs. That is, the number required is 94.1.

Suppose it is required to find the number of which the logarithm is 3.7936.

Look for 7936 in the table. It cannot be found, but the two adjacent mantissas between which it lies are seen to be 7931 and 7938; their difference is 7, and the difference between 7931 and 7936 is 5. Therefore, $\frac{5}{7}$ of the difference between the numbers corresponding to the mantissas, 7931 and 7938, must be added to the number corresponding to the mantissa 7931.

The number corresponding to the mantissa 7938 is 6220.

The number corresponding to the mantissa 7931 is 6210.

The difference between these numbers is 10,

and $6210 + \frac{5}{7} \text{ of } 10 = 6217.$

Therefore, the number required is 6217.

Suppose it is required to find the number of which the logarithm is 7.3882 — 10.

Look for 3882 in the table. It cannot be found, but the two adjacent mantissas between which it lies are seen to be 3874 and 3892; their difference is 18, and the difference between 3874 and 3882 is 8. Therefore, $\frac{8}{18}$ of the difference between the numbers corresponding to the mantissas, 3874 and 3892, must be added to the number corresponding to the mantissa 3874.

The number corresponding to the mantissa 3892 is 2450.

The number corresponding to the mantissa 3874 is 2440.

The difference between these numbers is 10,

and $2440 + \frac{8}{18} \text{ of } 10 = 2444.$

Therefore, the number required is 0.002444.

EXERCISE 112.

Find logarithms of the following numbers :

- | | | | |
|----------|------------|--------------|-------------|
| 1. 60. | 6. 3780. | 11. 70633. | 16. 877.08. |
| 2. 101. | 7. 54327. | 12. 12028. | 17. 73.896. |
| 3. 999. | 8. 90801. | 13. 0.00987. | 18. 7.0699. |
| 4. 9901. | 9. 10001. | 14. 0.87701. | 19. 0.0897. |
| 5. 5406. | 10. 10010. | 15. 1.0001. | 20. 99.778. |

Find antilogarithms to the following logarithms :

- | | | |
|-------------|------------------|------------------|
| 21. 4.2488. | 25. 4.7317. | 29. 9.0410 — 10. |
| 22. 3.6330. | 26. 1.9730. | 30. 9.8420 — 10. |
| 23. 2.5310. | 27. 9.8800 — 10. | 31. 7.0216 — 10. |
| 24. 1.9484. | 28. 0.2787. | 32. 8.6580 — 10. |

Ex. Find the product of $908.4 \times 0.05392 \times 2.117$.

$$\begin{array}{rcl}
 \log 908.4 & = & 2.9583 \\
 \log 0.05392 & = & 8.7318 - 10 \\
 \log 2.117 & = & 0.3257 \\
 \hline
 & & 2.0158 = \log 103.7. \text{ Ans.}
 \end{array}$$

Find by logarithms the following products :

- | | |
|---------------------------------|---------------------------------|
| 33. 948.76×0.043875 . | 35. 830.75×0.0003769 . |
| 34. 3.4097×0.0087634 . | 36. 8.4395×0.98274 . |

317. When any of the factors are *negative*, find their logarithms without regard to the signs; write the letter *n* after the logarithm that corresponds to a negative number. If the number of logarithms so marked is *odd*, the product is *negative*; if *even*, the product is positive.

Find the products of :

37. $7564 \times (-0.003764)$. 39. $-5.840359 \times (-0.00178)$.
 38. $3.7648 \times (-0.083497)$. 40. -8945.07×73.846 .

Ex. Find the quotient of $\frac{8.3709 \times 834.637}{7308.946}$.

$$\begin{aligned} \log 8.3709 &= 0.9227 \\ \log 834.637 &= 2.9215 \\ \text{colog } 7308.946 &= 6.1362 - 10 \\ 9.9804 - 10 &= \log 0.9558. \text{ Ans.} \end{aligned}$$

Find the quotients of :

41. $\frac{70654}{54013}$. 46. $\frac{0.07654}{83.947 \times 0.8395}$.
 42. $\frac{58706}{93078}$. 47. $\frac{7564 \times 0.07643}{8093 \times 0.09817}$.
 43. $\frac{8.32165}{0.07891}$. 48. $\frac{89 \times 753 \times 0.0097}{36709 \times 0.08497}$.
 44. $\frac{65039}{90761}$. 49. $\frac{413 \times 8.17 \times 3182}{915 \times 728 \times 2.315}$.
 45. $\frac{7.652}{-0.06875}$. 50. $\frac{212 \times (-6.12) \times (-2008)}{365 \times (-531) \times 2.576}$.

Ex. Find the cube of 0.0497.

$$\begin{aligned} \log 0.0497 &= 8.6964 - 10 \\ 3 \\ 6.0892 - 10 &= \log 0.0001228. \text{ Ans.} \end{aligned}$$

Find by logarithms :

51. 6.05^3 . 55. 0.78765^6 . 59. $(10\frac{2}{3})^4$. 63. $(3\frac{3}{4})^{4.17}$.
 52. 1.051^7 . 56. 0.691^9 . 60. $(1\frac{7}{9})^8$. 64. $(1\frac{2}{11})^{3.2}$.
 53. 1.1768^5 . 57. $(\frac{7}{8})^{11}$. 61. $(\frac{2}{3})^6$. 65. $(8\frac{3}{4})^{2.3}$.
 54. 1.3178^{10} . 58. $(\frac{1}{3})^7$. 62. $(7\frac{6}{11})^{0.38}$. 66. $(5\frac{3}{4})^{0.375}$.

Ex. Find the fourth root of 0.00862.

$$\log 0.00862 = 7.9355 - 10$$

$$\begin{array}{r} 30. \quad - 30 \\ 4 \overline{) 37.9355 - 40} \end{array}$$

$$9.4839 - 10 = \log 0.3047. \text{ Ans.}$$

Find by logarithms :

- | | | | |
|------------------------|---------------------------|-------------------------------------|--------------------------------------|
| 67. $7\frac{1}{2}$. | 70. $8379\frac{1}{2}$. | 73. $0.17643\frac{1}{2}$. | 76. $(\frac{71}{3408})\frac{1}{2}$. |
| 68. $11\frac{1}{2}$. | 71. $906.80\frac{1}{2}$. | 74. $2.5637\frac{1}{2}$. | 77. $(9\frac{1}{3})\frac{1}{2}$. |
| 69. $783\frac{1}{2}$. | 72. $8.1904\frac{1}{2}$. | 75. $(\frac{431}{88})\frac{1}{2}$. | 78. $(11\frac{1}{2})\frac{1}{2}$. |

Find by logarithms the values of :

$$79. \sqrt[5]{\frac{0.0075433^2 \times 78.343 \times 8172.4\frac{1}{2} \times 0.00052}{64285\frac{1}{2} \times 154.27^4 \times 0.001 \times 586.79\frac{1}{2}}}.$$

$$80. \sqrt[5]{\frac{15.832^3 \times 5793.6\frac{1}{2} \times 0.78426}{0.000327\frac{1}{2} \times 768.94^2 \times 3015.3 \times 0.007\frac{1}{2}}}.$$

$$81. \sqrt[5]{\frac{7.1895 \times 4764.2^2 \times 0.00326^5}{0.00048953 \times 457^3 \times 5764.4^2}}.$$

$$82. \sqrt[5]{\frac{3.1416 \times 4771.21 \times 2.7183\frac{1}{2}}{30.103^4 \times 0.4343\frac{1}{2} \times 69.897^4}}.$$

$$83. \sqrt[7]{\frac{0.03271^2 \times 53.429 \times 0.77542^3}{32.769 \times 0.000371^4}}.$$

$$84. \sqrt[3]{\frac{732.056^2 \times 0.0003572^4 \times 89793}{42.2798^3 \times 3.4574 \times 0.0026518^5}}.$$

$$85. \sqrt[3]{\frac{7932 \times 0.00657 \times 0.80464}{0.03274 \times 0.6428}}.$$

$$86. \sqrt[3]{\frac{7.1206 \times \sqrt{0.13274} \times 0.057389}{\sqrt{0.43468} \times 17.385 \times \sqrt{0.0096372}}}.$$

$$87. \left\{ \frac{3.075526^2 \times 5771.2\frac{1}{2} \times 0.0036984\frac{1}{2} \times 7.74}{72258 \times 327.93^3 \times 86.97^5} \right\}^{\frac{1}{2}}.$$

318. Since any positive number other than 1 may be taken as the base of a system of logarithms, the following general proofs to the base a should be noticed.

I. The logarithm of the product of two or more numbers is equal to the sum of the logarithms of the numbers.

For, let m and n be two numbers, and x and y their logarithms.

Then, by the definition of a logarithm, $m = a^x$, and $n = a^y$.

Hence, $m \times n = a^x \times a^y = a^{x+y}$.

$$\begin{aligned}\therefore \log (m \times n) &= x + y, \\ &= \log m + \log n.\end{aligned}$$

In like manner, the proposition may be extended to any number of factors.

II. The logarithm of a quotient is equal to the logarithm of the dividend minus the logarithm of the divisor.

For, let m and n be two numbers, and x and y their logarithms.

Then $m = a^x$, and $n = a^y$.

Hence, $m \div n = a^x \div a^y = a^{x-y}$.

$$\begin{aligned}\therefore \log (m \div n) &= x - y, \\ &= \log m - \log n.\end{aligned}$$

From this it follows that $\log \frac{1}{m} = \log 1 - \log m$.

But, since $\log 1 = 0$, $\log \frac{1}{m} = -\log m$.

III. The logarithm of a power of a number is equal to the logarithm of the number multiplied by the exponent of the power.

For, let x be the logarithm of m .

Then $m = a^x$,

and $m^p = (a^x)^p = a^{px}$.

$$\begin{aligned}\therefore \log m^p &= px, \\ &= p \log m.\end{aligned}$$

IV. *The logarithm of the root of a number is equal to the logarithm of the number divided by the index of the root.*

For, let x be the logarithm of m .

Then

$$m = a^x,$$

and

$$m^{\frac{1}{r}} = (a^x)^{\frac{1}{r}} = a^{\frac{x}{r}}.$$

$$\therefore \log m^{\frac{1}{r}} = \frac{x}{r} = \frac{\log m}{r}.$$

319. An **exponential equation**, that is, an equation in which the exponent is the unknown quantity, is easily solved by logarithms.

For, let

$$a^x = m.$$

Then

$$\log a^x = \log m.$$

$$\therefore x \log a = \log m.$$

$$\therefore x = \frac{\log m}{\log a}.$$

Ex. Find the value of x in $81^x = 10$.

$$81^x = 10,$$

$$x = \frac{\log 10}{\log 81}.$$

$$\therefore \log x = \log \log 10 + \text{colog } \log 81,$$

$$= 0 + 9.7193 - 10.$$

$$\therefore x = 0.524.$$

320. Logarithms of numbers to any base a may be converted into logarithms to any other base b by dividing the computed logarithms by the logarithm of b to the base a .

For, let

$$\log m = y$$

to the base b ,

and

$$\log b = x$$

to the base a .

Then

$$m = b^y, \text{ and } b = a^x.$$

$$\therefore m = (a^x)^y = a^{xy}.$$

$$\therefore \log m \text{ (to base } a) = xy = \log b \text{ (to base } a) \times \log m \text{ (to base } b).$$

$$\therefore \log m \text{ (to base } b) = \frac{\log m \text{ (to base } a)}{\log b \text{ (to base } a)}.$$

$$\text{This is usually written, } \log_b m = \frac{\log_a m}{\log_a b}.$$

CHAPTER XX.

RATIO, PROPORTION, AND VARIATION.

321. THE *relative magnitude* of two numbers is called their **ratio**, when expressed by the fraction which the first is of the second.

Thus, the ratio of 6 to 3 is indicated by the fraction $\frac{6}{3}$, which is sometimes written 6 : 3.

322. The first term of a ratio is called the **antecedent**, and the second term the **consequent**. When the antecedent is *equal* to the consequent, the ratio is called a *ratio of equality*; when the antecedent is *greater* than the consequent, the ratio is called a *ratio of greater inequality*; when *less*, a *ratio of less inequality*.

323. When the antecedent and consequent are interchanged, the resulting ratio is called the *inverse* of the given ratio.

Thus, the ratio 3 : 6 is the *inverse* of the ratio 6 : 3.

324. The ratio of two *quantities* that can be expressed in *integers* in terms of a *common unit* is equal to the ratio of the two *numbers* by which they are expressed.

Thus, the ratio of \$9 to \$11 is equal to the ratio of 9 : 11; and the ratio of a line $2\frac{1}{2}$ inches long to a line $3\frac{1}{2}$ inches long, when both are expressed in terms of a unit $\frac{1}{2}$ of an inch long, is equal to the ratio of 32 to 45.

325. Two quantities *different in kind* can have no ratio, for then one cannot be a fraction of the other.

326. Two quantities that can be expressed in integers in terms of a common unit are said to be *commensurable*. The common unit is called a *common measure*, and each quantity is called a *multiple* of this common measure.

Thus, a common measure of $2\frac{1}{2}$ feet and $3\frac{1}{2}$ feet is $\frac{1}{2}$ of a foot, which is contained 15 times in $2\frac{1}{2}$ feet, and 22 times in $3\frac{1}{2}$ feet. Hence, $2\frac{1}{2}$ feet and $3\frac{1}{2}$ feet are multiples of $\frac{1}{2}$ of a foot, $2\frac{1}{2}$ feet being obtained by taking $\frac{1}{2}$ of a foot 15 times, and $3\frac{1}{2}$ by taking $\frac{1}{2}$ of a foot 22 times.

327. When two quantities are *incommensurable*, that is, have no common unit in terms of which *both* quantities can be expressed in *integers*, it is impossible to find a fraction that will indicate the exact value of the ratio of the given quantities. It is possible, however, by taking the unit sufficiently small, to find a fraction that shall differ from the true value of the ratio by as little as we please.

Thus, if a and b denote the diagonal and side of a square,

$$\frac{a}{b} = \sqrt{2}.$$

Now $\sqrt{2} = 1.41421356.....$, a value greater than 1.414213, but less than 1.414214.

If, then, a *millionth part* of b be taken as the unit, the value of the ratio $\frac{a}{b}$ lies between $\frac{1414213}{1000000}$ and $\frac{1414214}{1000000}$, and therefore differs from either of these fractions by less than $\frac{1}{1000000}$.

By carrying the decimal farther, a fraction may be found that will differ from the true value of the ratio by less than a *billionth*, *trillionth*, or any other assigned value whatever.

328. Expressed generally, when a and b are incommensurable, and b is divided into any integral number (n) of equal parts, if one of these parts be contained in a more than m times, but less than $m+1$ times, then

$$\frac{a}{b} > \frac{m}{n}, \text{ but } < \frac{m+1}{n};$$

that is, the value of $\frac{a}{b}$ lies between $\frac{m}{n}$ and $\frac{m+1}{n}$.

The error, therefore, in taking either of these values for $\frac{a}{b}$ is $< \frac{1}{n}$. But by increasing n indefinitely, $\frac{1}{n}$ can be made to decrease indefinitely, and to become less than any assigned value, however small, though it cannot be made absolutely equal to zero.

329. The ratio between two incommensurable quantities is called an **incommensurable ratio**.

330. As the treatment of Proportion in Algebra depends upon the assumption that it is possible to find fractions which will *represent* the ratios, and as it appears that no fraction can be found to represent the exact value of an incommensurable ratio, it is necessary to show that *two incommensurable ratios are equal if their true values always lie between the same limits, however little these limits differ from each other*.

Let $a : b$ and $c : d$ be two incommensurable ratios.

Suppose the true values of the ratios $a : b$ and $c : d$ lie between $\frac{m}{n}$ and $\frac{m+1}{n}$. Then the *difference* between the true values of these ratios is *less* than $\frac{1}{n}$, however small the value of $\frac{1}{n}$ may be. § 328.

But since $\frac{1}{n}$ can be made to approach zero at pleasure, $\frac{1}{n}$ can be made *less* than *any assumed difference* between the ratios.

Therefore, to assume any difference between the ratios is to assume it possible to find a quantity that for the same value of $\frac{1}{n}$ shall be both *greater* and *less* than $\frac{1}{n}$; which is a manifest absurdity.

Hence, $a : b = c : d$.

331. It will be well to notice that the word **limit** means a fixed value from which another and variable value may be made to differ by as little as we please; it being impossible, however, for the difference between the variable value and the limit to become absolutely zero.

332. *A ratio will not be altered if both its terms be multiplied by the same number.*

For the ratio $a : b$ is represented by $\frac{a}{b}$, the ratio $ma : mb$ is represented by $\frac{ma}{mb}$; and since $\frac{ma}{mb} = \frac{a}{b}$, $\therefore ma : mb = a : b$.

333. *A ratio will be altered if different multipliers of its terms be taken; and will be increased or diminished according as the multiplier of the antecedent is greater or less than that of the consequent.*

For, $ma : nb$ will be $>$ or $<$ $a : b$
 according as $\frac{ma}{nb}$ is $>$ or $<$ $\frac{a}{b}$ ($= \frac{na}{nb}$)
 as ma is $>$ or $<$ na ,
 as m is $>$ or $<$ n .

334. *A ratio of greater inequality will be diminished, and a ratio of less inequality increased by adding the same number to both its terms.*

For, $a + x : b + x$ is $>$ or $<$ $a : b$
 according as $\frac{a + x}{b + x}$ is $>$ or $<$ $\frac{a}{b}$,
 as $ab + bx$ is $>$ or $<$ $ab + ax$,
 as bx is $>$ or $<$ ax ,
 as b is $>$ or $<$ a .

335. *A ratio of greater inequality will be increased, and a ratio of less inequality diminished, by subtracting the same number from both its terms.*

For, $a - x : b - x$ will be $>$ or $<$ $a : b$
 according as $\frac{a - x}{b - x}$ is $>$ or $<$ $\frac{a}{b}$,
 as $ab - bx$ is $>$ or $<$ $ab - ax$,
 as ax is $>$ or $<$ bx ,
 as a is $>$ or $<$ b .

336. Ratios are *compounded* by taking the product of the fractions that represent them.

Thus, the ratio compounded of $a:b$ and $c:d$ is found by taking the product of $\frac{a}{b}$ and $\frac{c}{d} = \frac{ac}{bd}$.

The ratio compounded of $a:b$ and $a:b$ is the *duplicate* ratio $a^2:b^2$, and the ratio compounded of $a:b$, $a:b$, and $a:b$ is the *triplicate* ratio $a^3:b^3$.

337. Ratios are *compared* by comparing the fractions that represent them.

Thus,	$a:b$ is $>$ or $<$ $c:d$
according as	$\frac{a}{b}$ is $>$ or $<$ $\frac{c}{d}$,
as	$\frac{ad}{bd}$ is $>$ or $<$ $\frac{bc}{bd}$,
as	ad is $>$ or $<$ bc .

EXERCISE 113.

- Write down the ratio compounded of $3:5$ and $8:7$. Which of these ratios is increased, and which is diminished by the composition?
- Compound the duplicate ratio of $4:15$ with the triplicate of $5:2$.
- Show that a duplicate ratio is greater or less than its simple ratio according as it is a ratio of greater or less inequality.
- Arrange in order of magnitude the ratios $3:4$; $23:25$; $10:11$; and $15:16$.
- Arrange in order of magnitude $a+b:a-b$ and $a^2+b^2:a^2-b^2$, if $a > b$.

Find the ratios compounded of:

- $3:5$; $10:21$; $14:15$.
- $7:9$; $102:105$; $15:17$.

8. $\frac{a^2 + ax + x^2}{a^3 - a^2x + ax^2 - x^3}$ and $\frac{a^2 - ax + x^2}{a + x}$.
9. $\frac{x^2 - 9x + 20}{x^2 - 6x}$ and $\frac{x^2 - 13x + 42}{x^2 - 5x}$.
10. $a + b : a - b$; $a^2 + b^2 : (a + b)^2$; $(a^2 - b^2)^2 : a^4 - b^4$.
11. Two numbers are in the ratio 2 : 3, and if 9 is added to each, they are in the ratio 3 : 4. Find the numbers.
(Let $2x$ and $3x$ represent the numbers.)
12. Show that the ratio $a : b$ is the duplicate of the ratio $a + c : b + c$, if $c^2 = ab$.
13. Find two numbers in the ratio 3 : 4, of which the sum is to the sum of their squares in the ratio of 7 to 50.
14. If five gold coins and four silver ones are worth as much as three gold coins and twelve silver ones, find the ratio of the value of a gold coin to that of a silver one.
15. If eight gold and nine silver coins are worth as much as six gold and nineteen silver coins, find the ratio of the value of a silver coin to that of a gold one.
16. There are two roads from A to B, one of them 14 miles longer than the other; and two roads from B to C, one of them 8 miles longer than the other. The distance from A to B is to the distance from B to C, by the shorter roads, as 1 to 2; by the longer roads, as 2 to 3. Find the distances.
17. What must be added to each of the terms of the ratio $m : n$, that it may become equal to the ratio $p : q$?
18. A rectangular field contains 5270 acres, and its length is to its breadth in the ratio of 31 : 17. Find its dimensions.

PROPORTION.

338. An equation consisting of two equal ratios is called a **proportion**; and the terms of the ratios are called **proportionals**.

339. The algebraic test of a proportion is that the two fractions which represent the ratios shall be equal.

Thus, the ratio $a:b$ will be equal to the ratio $c:d$ if $\frac{a}{b} = \frac{c}{d}$; and the four numbers a, b, c, d are called **proportionals**, or are said to be in proportion.

340. If the ratios $a:b$ and $c:d$ form a proportion, the proportion is written $a:b=c:d$, and is read the ratio of a to b is equal to the ratio of c to d ;
or $a:b::c:d$, and is read a is to b in the same ratio as c is to d .

The first and last terms, a and d , are called the **extremes**.

The two middle terms, b and c , are called the **means**.

341. *When four numbers are in proportion, the product of the extremes is equal to the product of the means.*

For, if $a:b::c:d$,

then $\frac{a}{b} = \frac{c}{d}$.

By multiplying by bd , $ad = bc$.

342. *If the product of two numbers is equal to the product of two others, either two may be made the extremes of a proportion and the other two the means.*

For, if $ad = bc$,

by dividing by bd , $\frac{ad}{bd} = \frac{bc}{bd}$,

or $\frac{a}{b} = \frac{c}{d}$.

$\therefore a:b::c:d$.

343. The equation $ad = bc$ gives

$$a = \frac{bc}{d}; \quad b = \frac{ad}{c};$$

so that an extreme may be found by dividing the product of the means by the other extreme; and a mean may be found by dividing the product of the extremes by the other mean.

344. If four quantities, a, b, c, d , be in proportion, they will be in proportion by :

I. Inversion.

That is, b will be to a as d is to c .

$$\begin{aligned} \text{For, if} \quad & a : b :: c : d, \\ \text{then} \quad & \frac{a}{b} = \frac{c}{d}, \\ \text{and} \quad & 1 \div \frac{a}{b} = 1 \div \frac{c}{d}, \\ \text{or} \quad & \frac{b}{a} = \frac{d}{c}. \\ & \therefore b : a :: d : c. \end{aligned}$$

345. II. Composition.

That is, $a + b$ will be to b as $c + d$ is to d .

$$\begin{aligned} \text{For, if} \quad & a : b :: c : d, \\ \text{then} \quad & \frac{a}{b} = \frac{c}{d}, \\ \text{and} \quad & \frac{a}{b} + 1 = \frac{c}{d} + 1, \\ \text{or} \quad & \frac{a + b}{b} = \frac{c + d}{d}. \\ & \therefore a + b : b :: c + d : d. \end{aligned}$$

346. III. Division.

That is, $a - b$ will be to b as $c - d$ is to d .

$$\begin{aligned} \text{For, if} \quad & a : b :: c : d, \\ \text{then} \quad & \frac{a}{b} = \frac{c}{d}, \end{aligned}$$

$$\text{and} \quad \frac{a}{b} - 1 = \frac{c}{d} - 1,$$

$$\text{or} \quad \frac{a-b}{b} = \frac{c-d}{d}.$$

$$\therefore a-b : b :: c-d : d.$$

347. IV. Composition and Division.

That is, $a+b$ will be to $a-b$ as $c+d$ is to $c-d$.

$$\text{For, from II.,} \quad \frac{a+b}{b} = \frac{c+d}{d},$$

$$\text{and from III.,} \quad \frac{a-b}{b} = \frac{c-d}{d}.$$

$$\text{By dividing,} \quad \frac{a+b}{a-b} = \frac{c+d}{c-d}.$$

$$\therefore a+b : a-b :: c+d : c-d.$$

348. When the four quantities a, b, c, d are all of the *same kind*, they will be in proportion by :

V. Alternation.

That is, a will be to c as b is to d .

$$\text{For, if} \quad a : b :: c : d,$$

$$\text{then} \quad \frac{a}{b} = \frac{c}{d}.$$

$$\text{By multiplying by } \frac{b}{c}, \quad \frac{ab}{bc} = \frac{bc}{cd},$$

$$\text{or} \quad \frac{a}{c} = \frac{b}{d}.$$

$$\therefore a : c :: b : d.$$

349. From the proportion $a : c :: b : d$ may be obtained by :

$$\text{VI. Composition.} \quad a+c : c :: b+d : d.$$

$$\text{VII. Division.} \quad a-c : c :: b-d : d.$$

$$\text{VIII. Composition and Division.} \quad a+c : a-c :: b+d : b-d.$$

350. *In a series of equal ratios, the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.*

For, if $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \frac{g}{h},$

r may be put for each of these ratios.

Then $\frac{a}{b} = r, \frac{c}{d} = r, \frac{e}{f} = r, \frac{g}{h} = r.$

$$\therefore a = br, c = dr, e = fr, g = hr.$$

$$\therefore a + c + e + g = (b + d + f + h)r.$$

$$\therefore \frac{a + c + e + g}{b + d + f + h} = r = \frac{a}{b}.$$

$$\therefore a + c + e + g : b + d + f + h :: a : b.$$

In like manner it may be shown that

$$ma + nc + pe + qg : mb + nd + pf + qh :: a : b.$$

351. If a, b, c, d be in *continued* proportion, that is, if $a : b = b : c = c : d$, then will $a : c = a^2 : b^2$ and $a : d = a^3 : b^3$.

For, $\frac{a}{b} = \frac{b}{c} = \frac{c}{d}.$

Hence, $\frac{a}{b} \times \frac{b}{c} = \frac{a}{b} \times \frac{a}{b},$

or $\frac{a}{c} = \frac{a^2}{b^2}.$

$$\therefore a : c = a^2 : b^2.$$

So $\frac{a}{b} \times \frac{b}{c} \times \frac{c}{d} = \frac{a}{b} \times \frac{a}{b} \times \frac{a}{b},$

or $\frac{a}{d} = \frac{a^3}{b^3}.$

$$\therefore a : d = a^3 : b^3.$$

352. If a, b, c are proportionals, so that $a : b :: b : c$, then b is called a *mean* proportional between a and c , and c is called a *third* proportional to a and b .

If $a : b :: b : c$, then $b = \sqrt{ac}$.

$$\begin{array}{ll} \text{For, if} & a : b :: b : c, \\ \text{then} & \frac{a}{b} = \frac{b}{c}, \\ \text{and} & b^2 = ac. \\ \therefore & b = \sqrt{ac}. \end{array}$$

353. *The products of the corresponding terms of two or more proportions are in proportion.*

$$\begin{array}{ll} \text{For, if} & a : b :: c : d, \\ & e : f :: g : h, \\ \text{and} & k : l :: m : n, \\ \text{then} & \frac{a}{b} = \frac{c}{d}, \frac{e}{f} = \frac{g}{h}, \frac{k}{l} = \frac{m}{n}. \end{array}$$

Hence, by finding the product of the left members, and also of the right members of these equations,

$$\begin{array}{l} \frac{aek}{bfl} = \frac{cgm}{dhn}. \\ \therefore aek : bfl :: cgm : dhn. \end{array}$$

354. *Like powers, or like roots, of the terms of a proportion are in proportion.*

$$\begin{array}{ll} \text{For, if} & a : b :: c : d, \\ \text{then} & \frac{a}{b} = \frac{c}{d}. \end{array}$$

By raising both sides to the n th power.

$$\begin{array}{l} \frac{a^n}{b^n} = \frac{c^n}{d^n}. \\ \therefore a^n : b^n :: c^n : d^n. \end{array}$$

By extracting the n th root,

$$\begin{array}{l} \frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}} = \frac{c^{\frac{1}{n}}}{d^{\frac{1}{n}}}. \\ \therefore a^{\frac{1}{n}} : b^{\frac{1}{n}} :: c^{\frac{1}{n}} : d^{\frac{1}{n}}. \end{array}$$

355. *If two quantities be increased or diminished by like parts of each, the results will be in the same ratio as the quantities themselves.*

$$\begin{aligned} \text{For,} \quad \frac{a}{b} &= \frac{\left(1 \pm \frac{m}{n}\right)a}{\left(1 \pm \frac{m}{n}\right)b}, \\ \text{that is,} \quad \frac{a}{b} &= \frac{a \pm \frac{m}{n}a}{b \pm \frac{m}{n}b}. \\ \therefore a : b &:: a \pm \frac{m}{n}a : b \pm \frac{m}{n}b. \end{aligned}$$

356. The laws that have been established for ratios should be remembered when ratios are expressed in their fractional form.

(1) Solve:
$$\frac{x^2 + x + 1}{x^2 - x - 1} = \frac{x^2 - x + 2}{x^2 + x - 2}.$$

By § 347
$$\frac{2x^2}{2(x+1)} = \frac{2x^2}{-2(x-2)},$$

 and this equation is satisfied, when $x = 0$;
 or, dividing by $\frac{2x^2}{2}$,
$$\frac{1}{x+1} = \frac{1}{2-x}.$$

$$\therefore x = \frac{1}{2}.$$

(2) If $a : b :: c : d$, show that

$$a^2 + ab : b^2 - ab :: c^2 + cd : d^2 - cd.$$

If
$$\frac{a}{b} = \frac{c}{d},$$

 then
$$\frac{a+b}{a-b} = \frac{c+d}{c-d}, \quad \text{§ 347.}$$

 and
$$\frac{a}{-b} = \frac{c}{-d}.$$

$$\therefore \frac{a}{-b} \times \frac{a+b}{a-b} = \frac{c}{-d} \times \frac{c+d}{c-d}; \quad \text{§ 353.}$$

 that is,
$$\frac{a^2 + ab}{b^2 - ab} = \frac{c^2 + cd}{d^2 - cd},$$

 or
$$a^2 + ab : b^2 - ab :: c^2 + cd : d^2 - cd.$$

- (3) When $a:b::c:d$, and a is the *greatest term*, show that $a+d$ is greater than $b+c$.

$$\begin{array}{ll}
 \text{Since} & \frac{a}{b} = \frac{c}{d}, \text{ and } a > c, \\
 \text{then} & b > d. \\
 \text{Also, since} & \frac{a-b}{b} = \frac{c-d}{d}, \quad \S 346. \\
 \text{and} & b > d, \\
 \text{then} & a-b > c-d. \\
 \text{By adding to this,} & b+d = b+d, \\
 & a+d > b+c.
 \end{array}$$

EXERCISE 114.

If $a:b::c:d$, prove that:

1. $ma:nb::mc:nd$.
2. $3a+b:b::3c+d:d$.
3. $a+2b:b::c+2d:d$.
4. $a^3:b^3::c^3:d^3$.
5. $a:a+b::c:c+d$.
6. $a:a-b::c:c-d$.
7. $ma+nb:ma-nb::mc+nd:mc-nd$.
8. $2a+3b:3a-4b::2c+3d:3c-4d$.
9. $ma^2+nc^2:mb^2+nd^2::a^2:b^2$.
10. $ma^2+nab+pb^2:mc^2+ncd+pd^2::b^2:d^2$.

If $a:b::b:c$, prove that:

11. $a+b:b+c::a:b$.
12. $a^2+ab:b^2+bc::a:c$.
13. $a:c::(a+b)^2:(b+c)^2$.
14. When a, b , and c are proportionals, and a the greatest, show that $a+c > 2b$.
15. If $\frac{x-y}{l} = \frac{y-z}{m} = \frac{z-x}{n}$, and x, y, z are unequal, show that $l+m+n=0$.

16. Find x when $x+5:2x-3::5x+1:3x-3$.
17. Find x when $x+a:2x-b::3x+b:4x-a$.
18. Find x when $\sqrt{x}+\sqrt{b}:\sqrt{x}-\sqrt{b}::a:b$.
19. Find x and y when $x:27::y:9$, and $x:27::2:x-y$.
20. Find x and y when $x+y+1:x+y+2::6:7$, and
when $y+2x:y-2x::12x+6y-3:6y-12x-1$.
21. Find x when $x^2-4x+2:x^2-2x-1::x^2-4x:x^2-2x-2$.
22. A railway passenger observes that a train passes him, moving in the opposite direction, in 2 seconds; but moving in the same direction with him, it passes him in 30 seconds. Compare the rates of the two trains.
23. A and B trade with different sums. A gains \$200 and B loses \$50, and now A's stock:B's::2: $\frac{1}{2}$. But, if A had gained \$100 and B lost \$85, their stocks would have been as 15: $3\frac{1}{4}$. Find the original stock of each.
24. A quantity of milk is increased by watering in the ratio 4:5, and then three gallons are sold; the remainder is mixed with three quarts of water, and is increased in the ratio 6:7. How many gallons of milk were there at first?
25. In a mile race between a bicycle and a tricycle their rates were as 5:4. The tricycle had half a minute start, but was beaten by 176 yards. Find the rate of each.
26. The time which an express-train takes to travel 180 miles is to that taken by an ordinary train as 9:14. The ordinary train loses as much time from stopping as it would take to travel 30 miles; the express-train loses only half as much time as the other by stopping, and travels 15 miles an hour faster. What are their respective rates?

27. A line is divided into two parts in the ratio 2:3, and into two parts in the ratio 3:4; the distance between the points of section is 2. Find the length of the line.
28. A railway consists of two sections; the annual expenditure on one is increased this year 5%, and on the other 4%, producing on the whole an increase of $4\frac{3}{4}\%$. Compare the amount expended on the two sections last year, and also this year.
29. When a, b, c, d are proportional and unequal, show that no number x can be found such that $a+x, b+x, c+x, d+x$ shall be proportionals.

VARIATION.

357. Two quantities may be so related that, when one has its value changed, the other will, in consequence, have its value changed.

Thus, the *distance* travelled in a certain time will be *doubled* if the *rate* be *doubled*. The *time* required for doing a certain quantity of work will be *doubled* if only *half* the number of *workmen* be employed.

358. Whenever it becomes necessary to express the *general relations* of certain kinds of quantities to each other, without confining the inquiry to any *particular values* of these quantities, it will usually be sufficient to mention two of the terms of a proportion. In all such cases, however, four terms are always implied.

Thus, if it be said that the weight of water is proportional to its volume, or varies as its volume, the meaning is, that *one* gallon of water is to *any number* of gallons as the weight of *one* gallon is to the weight of the *given number* of gallons.

359. Quantities used in a *general* sense, as distance, time, weight, volume, to which particular values may be assigned, are denoted by capital letters, A, B, C , etc. ; while *assigned values* of these quantities may be denoted by small letters, a, b, c , etc. The letters A, B, C will be understood to represent *any numerical values* that may be assigned to the quantities ; and when two such letters occur in an expression they will be understood to represent *any corresponding* numerical values that may be assigned to the two quantities.

360. When two quantities A and B are so connected that their ratio is *constant*, that is, remains the same for all corresponding values of A and B , the one is said to *vary as* the other ; and this relation is expressed by $A \propto B$ (read *A varies as B*).

Thus, the area of a triangle with a given base *varies as* its altitude ; for, if the altitude be changed, the area will be changed in the same ratio.

If this constant ratio be denoted by m , then $\frac{A}{B} = m$, or $A = mB$.

From this equation m may be found when two corresponding values of A and B are known.

361. When two quantities are so connected that if one be changed in any ratio, the other will be changed in the *inverse* ratio, the one is said to *vary inversely as* the other.

Thus, the time required to do a certain amount of work *varies inversely as* the number of workmen employed ; for, if the number of workmen be doubled, halved, or changed in any ratio, the time required will be halved, doubled, or changed in the inverse ratio.

362. If A varies inversely as B , two values of A have to each other the inverse ratio of the two corresponding values of B ; or $a : a' :: b' : b$, that is, $ab = a'b'$.

Hence, the product AB is constant, and may be denoted by m . That is, $AB = m$.

If any two corresponding values of A and B are known, the constant m may be found.

The equation $AB = m$ may be written $A = \frac{m}{B}$, and as m is constant, A is said to vary as the *reciprocal* of B , or $A \propto \frac{1}{B}$.

363. The two equations,

$$A = mB \text{ (for direct variation),}$$

$$A = \frac{m}{B} \text{ (for inverse variation),}$$

furnish the simplest method of treating Variation.

If $A = mBC$, A is said to vary *jointly* as B and C .

If $A = \frac{mB}{C}$, A is said to vary *directly* as B and *inversely* as C .

364. The following results are to be observed :

I. If $A \propto B$ and $B \propto C$, then $A \propto C$.

$$\begin{aligned} \text{For} \quad & A = mB, \text{ where } m \text{ is constant,} \\ \text{and} \quad & B = nC, \text{ where } n \text{ is constant.} \\ \therefore & A = mnC. \\ \therefore & A \propto C, \text{ since } mn \text{ is constant.} \end{aligned}$$

In like manner, if $A \propto B$ and $B \propto \frac{1}{C}$, then $A \propto \frac{1}{C}$.

II. If $A \propto C$ and $B \propto C$, then $A \pm B \propto C$, and $\sqrt{AB} \propto C$.

$$\begin{aligned} \text{For} \quad & A = mC, \text{ where } m \text{ is constant,} \\ \text{and} \quad & B = nC, \text{ where } n \text{ is constant.} \\ \therefore & A \pm B = (m \pm n) C. \\ \therefore & A \pm B \propto C, \text{ since } m \pm n \text{ is constant.} \end{aligned}$$

$$\begin{aligned} \text{Also,} \quad & \sqrt{AB} = \sqrt{mC \times nC} = \sqrt{mn}C = C \sqrt{mn}. \\ \therefore & \sqrt{AB} \propto C, \text{ since } \sqrt{mn} \text{ is constant.} \end{aligned}$$

III. If $A \propto B$ and $C \propto D$, then $AC \propto BD$.

For $A = mB$, where m is constant,
 $C = nD$, where n is constant.
 $\therefore AC = mnBD$.
 $\therefore AC \propto BD$, since mn is constant.

IV. If $A \propto B$ then $A^n \propto B^n$.

For $A = mB$, where m is constant.
 $\therefore A^n = m^n B^n$.
 $\therefore A^n \propto B^n$, since m^n is constant.

V. If $A \propto B$ when C is unchanged, and $A \propto C$ when B is unchanged, then $A \propto BC$ when both B and C change.

For $A = mB$, when B varies and C is constant.
 Here, m is *constant* and cannot contain the *variable* B .
 $\therefore A$ must contain B , but no other power of B .
 Again, $A = nC$, when C varies and B is constant.
 Here, n is *constant* and cannot contain the *variable* C ,
 $\therefore A$ must contain C , but no other power of C .
 Hence, A contains both B and C , but no other powers of B and C , and therefore,

$$\frac{A}{BC} = p, \text{ or } A = pBC, \text{ where } p \text{ is constant.}$$

$$\therefore A \propto BC, \text{ since } p \text{ is constant.}$$

In like manner, it may be shown that if A varies as each of any number of quantities B, C, D , etc., when the rest are unchanged, then when they all change, $A \propto BCD$, etc.

Thus, the area of a rectangle varies as the base when the altitude is constant, and as the altitude when the base is constant, but as the product of the base and altitude when both vary.

The volume of a rectangular solid varies as the length when the width and thickness remain constant; as the width when the length and thickness remain constant; as the thickness when the length and width remain constant; but as the product of length, breadth, and thickness when all three vary.

- (1) If A varies inversely as B , and when $A=2$ the corresponding value of B is 36, find the corresponding value of B when $A=9$.

$$\begin{aligned}\text{Here } A &= \frac{m}{B}, \\ \text{or } m &= AB. \\ \therefore m &= 2 \times 36 = 72.\end{aligned}$$

And if 9 and 72 are substituted for A and m respectively in

$$\begin{aligned}A &= \frac{m}{B}, \\ \text{the result is } 9 &= \frac{72}{B}. \\ \therefore 9B &= 72. \\ \therefore B &= 8. \text{ Ans.}\end{aligned}$$

- (2) The weight of a sphere of given material varies as its volume, and its volume varies as the cube of its diameter. If a sphere 4 inches in diameter weighs 20 pounds, find the weight of a sphere 5 inches in diameter.

Let W represent the weight,
 V represent the volume,
 D represent the diameter.

Then $W \propto V$ and $V \propto D^3$.

$$\therefore W \propto D^3$$

Put $W = mD^3$,

then, since 20 and 4 are corresponding volumes of W and D ,

$$\begin{aligned}20 &= m \times 64, \\ \therefore m &= \frac{20}{64} = \frac{5}{16}. \\ \therefore W &= \frac{5}{16} D^3. \\ \therefore \text{when } D &= 5, W = \frac{5}{16} \text{ of } 125 = 39\frac{1}{16}.\end{aligned}$$

EXERCISE 115.

1. If $A \propto B$, and $A=4$ when $B=5$, find A when $B=12$.
2. If $A \propto B$, and when $B=\frac{1}{2}$, $A=\frac{1}{3}$, find A when $B=\frac{1}{8}$.
3. If A varies jointly as B and C , and 3, 4, 5 are simultaneous values of A , B , C , find A when $B=C=10$.

4. If $A \propto \frac{1}{B}$, and $B=2$, when $A=10$, find the value of B when $A=4$.
5. If $A \propto \frac{B}{C}$, and $B=4$, and $C=3$, when $A=6$, find the value of A when $B=5$ and $C=7$.
6. If the square of X varies as the cube of Y , and $X=3$ when $Y=4$, find the equation between X and Y .
7. If the square of X varies inversely as the cube of Y , and $X=2$ when $Y=3$, find the equation between X and Y .
8. If Z varies as X directly and Y inversely, and if $X=3$, and $Y=4$, when $Z=2$, find the value of Z when $X=15$ and $Y=8$.
9. If $A \propto B + c$ where c is constant, and if $A=2$ when $B=1$, and if $A=5$ when $B=2$, find A when $B=3$.
10. The velocity acquired by a stone falling from rest varies as the time of falling; and the distance fallen varies as the square of the time. If it is found that in 3 seconds a stone has fallen 145 feet, and acquired a velocity of $96\frac{2}{3}$ feet per second, find the velocity and distance at the end of 5 seconds.
11. If a heavier weight draw up a lighter one by means of a string passing over a fixed wheel, the space described in a given time will vary directly as the difference between the weights, and inversely as their sum. If 9 ounces draw 7 ounces through 8 feet in 2 seconds, how high will 12 ounces draw 9 ounces in the same time?
12. The space will vary also as the square of the time. Find the space in Example 11, if the time in the latter case is 3 seconds.

13. Equal volumes of iron and copper are found to weigh 77 and 89 ounces respectively. Find the weight of $10\frac{1}{2}$ feet of round copper rod when 9 inches of iron rod of the same diameter weigh $31\frac{9}{16}$ ounces.
14. The square of the time of a planet's revolution varies as the cube of its distance from the sun. The distances of the Earth and Mercury from the sun being 91 and 35 millions of miles, find in days the time of Mercury's revolution.
15. A spherical iron shell 1 foot in diameter weighs $\frac{91}{16}$ of what it would weigh if solid. Find the thickness of the metal, it being known that the volume of a sphere varies as the cube of its diameter.
16. The volume of a sphere varies as the cube of its diameter. Compare the volume of a sphere 6 inches in diameter with the sum of the volumes of 3 spheres whose diameters are 3, 4, 5 inches respectively.
17. Two circular gold plates, each an inch thick, the diameters of which are 6 inches and 8 inches respectively, are melted and formed into a single circular plate 1 inch thick. Find its diameter, having given that the area of a circle varies as the square of its diameter.
18. The volume of a pyramid varies jointly as the area of its base and its altitude. A pyramid, the base of which is 9 feet square, and the height of which is 10 feet, is found to contain 10 cubic yards. What must be the height of a pyramid upon a base 3 feet square, in order that it may contain 2 cubic yards?

CHAPTER XXI.

SERIES.

365. A succession of numbers which proceed according to some fixed law is called a **series**; and the successive numbers are called the **terms** of the series.

Thus, by executing the indicated division of $\frac{1}{1-x}$, the series $1+x+x^2+x^3+\dots$ is obtained, a series that has an *unlimited* number of terms.

366. A series that is continued *indefinitely* is called an **infinite series**; and a series that comes to an end at some particular term is called a **finite series**.

367. When x is less than 1, the more terms we take of the infinite series $1+x+x^2+x^3+\dots$, obtained by dividing 1 by $1-x$, the more nearly does their sum *approach* to the value of $\frac{1}{1-x}$.

Thus, if $x = \frac{1}{2}$, then $\frac{1}{1-x} = \frac{1}{1-\frac{1}{2}} = \frac{2}{1} = 2$, and the series becomes $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$, a sum which cannot become equal to 2 however great the number of terms taken, but which may be made to differ from 2 by as little as we please by increasing indefinitely the number of terms.

368. But when x is greater than 1, the more terms we take of the series $1+x+x^2+x^3+\dots$ the more does the sum of the series *diverge* from the value of $\frac{1}{1-x}$.

Thus, if $x = 3$, then $\frac{1}{1-x} = \frac{1}{1-3} = -\frac{1}{2}$, and the series becomes $1 + 3 + 9 + 27 + \dots$, a sum which *diverges* more and more from $-\frac{1}{2}$,

the more terms we take, and which may be made to increase indefinitely by increasing indefinitely the number of terms taken.

369. A series whose sum as the number of its terms is indefinitely increased approaches some *fixed finite value as a limit* is called a **converging series**; and a series whose sum increases indefinitely as the number of its terms is increased, is called a **diverging series**.

370. When $x=1$, the division of 1 by $1-x$, that is, of 1 by 0, has no meaning, according to the *definition of division*; and any attempt to divide by a divisor that is equal to zero leads to absurd results.

Thus, $8+4=8+4$;
by transposing, $8-8=4-4$;
or, dividing by $4-4$, $2=1$; a manifest absurdity.

371. When $x=1$ *very nearly*, then the value of $\frac{1}{1-x}$ will be *very great*, and the sum of the series $1+x+x^2+x^3+\dots$ will become greater and greater the more terms we take. Hence, by making the denominator $1-x$ approach indefinitely to zero, the value of the fraction $\frac{1}{1-x}$ may be made to increase at pleasure.

372. If the symbol \circ be used to denote a number that is less than any assignable number, and that may be considered to decrease without limit, not, however, becoming 0, and the symbol ∞ be used to denote a number that is greater than any assignable number, and that may be considered to increase without limit, not, however, becoming ∞ , then

$$\frac{1}{\circ} = \infty.$$

In the same sense $\frac{a}{\circ} = \infty$, where a represents any value that may be assigned.

373. If x in the fraction $\frac{1-x^n}{1-x}$ be equal to 1, the numerator and denominator will each become 0, and the fraction will assume the form $\frac{0}{0}$.

374. If, however, x in this fraction approach to 1 as its limit, then the denominator $1-x$, inasmuch as it has *some* value, even though less than any assignable value, may be used as a divisor, and the result is $1+x+x^2+x^3+x^4$. Hence, it is evident that though both terms of the fraction become smaller and smaller as $1-x$ approaches to 0, still the numerator becomes more and more nearly *five times* the denominator.

It may be remarked that when the symbol $\frac{0}{0}$ is obtained for the value of the unknown number in a problem, the meaning is that the problem has no *definite* solution, but that its conditions are satisfied if any value whatever be taken for the required number; and if the symbol $\frac{a}{0}$, in which a denotes any assigned value, is obtained for the value of the unknown number, the meaning is that the conditions of the problem are impossible.

375. The number of different series is unlimited, but the only kind of series that will be considered at this stage of the work are Arithmetical, Geometrical, and Harmonical Series.

ARITHMETICAL SERIES.

376. A series in which the difference between any two adjacent terms is equal to the difference between any other two adjacent terms, is called an **Arithmetical Series** or an **Arithmetical Progression**.

377. The general representative of such a series will be

$$a, a+d, a+2d, a+3d, \dots,$$

in which a is the first term and d the common difference;

and the series will be *increasing* or *decreasing* according as d is positive or negative.

378. Since each succeeding term of the series is obtained by adding d to the preceding term, the coefficient of d will always be 1 less than the number of the term, so that the

$$nth \text{ term} = a + (n-1)d.$$

If the nth term is denoted by l , this equation becomes

$$l = a + (n-1)d. \quad (1)$$

379. The *arithmetical mean* between two numbers is the number which stands between them, and makes with them an arithmetical series.

380. If a and b denote two numbers, and A their arithmetical mean, then, by the definition of an arithmetical series,

$$\begin{aligned} A - a &= b - A. \\ \therefore A &= \frac{a+b}{2}. \end{aligned} \quad (2)$$

381. Sometimes it is required to insert several arithmetical means between two numbers.

If m = the number of means, then $m+2=n$, the whole number of terms; and if $m+2$ is substituted for n in the equation

$$\begin{aligned} l &= a + (n-1)d, \\ \text{the result is} \quad l &= a + (m+1)d. \\ \text{By transposing } a, \quad l - a &= (m+1)d. \\ \therefore \frac{l-a}{m+1} &= d. \end{aligned} \quad (3)$$

Thus, if it is required to insert six means between 3 and 17, the value of d is found to be $\frac{17-3}{6+1} = 2$; and the series will be 3, 5, 7, 9, 11, 13, 15, 17.

382. If l denotes the last term, a the first term, n the number of terms, d the common difference, and s the sum of the terms, it is evident that

$$\begin{aligned} s &= a + (a+d) + (a+2d) + \cdots + (l-d) + l, \text{ or} \\ s &= l + (l-d) + (l-2d) + \cdots + (a+d) + a \\ \therefore 2s &= (a+l) + (a+l) + (a+l) + \cdots + (a+l) + (a+l) \\ &= n(a+l) \\ \therefore s &= \frac{n}{2}(a+l). \end{aligned} \quad (4)$$

383. From the two equations,

$$l = a + (n-1)d, \quad (1)$$

$$s = \frac{n}{2}(a+l), \quad (2)$$

any one of the numbers a , d , l , n , s may be found when *three* are given.

Ex. Find n when d , l , s are given.

$$\text{From (1),} \quad a = l - (n-1)d.$$

$$\text{From (2),} \quad a = \frac{2s - ln}{n}.$$

$$\text{Therefore, } l - (n-1)d = \frac{2s - ln}{n}.$$

$$\therefore ln - dn^2 + dn = 2s - ln.$$

$$\therefore dn^2 - (2l + d)n = -2s.$$

$$\therefore 4d^2n^2 - (2l + d)^2 = (2l + d)^2 - 8ds,$$

$$\therefore 2dn - (2l + d) = \pm \sqrt{(2l + d)^2 - 8ds}.$$

$$\therefore n = \frac{2l + d \pm \sqrt{(2l + d)^2 - 8ds}}{2d}.$$

NOTE. The table on the following page contains the results of the general solution of all possible problems in arithmetical series. The student is advised to work these out, both for the results obtained and for the practice gained in solving literal equations in which the unknown numbers are represented by other letters than x , y , z .

No.	GIVEN.	REQUIRED.	RESULTS.
1	$a d n$	l	$l = a + (n-1) d.$
2	$a d s$		$l = -\frac{1}{2}d \pm \sqrt{2ds + (a - \frac{1}{2}d)^2}.$
3	$a n s$		$l = \frac{2s}{n} - a.$
4	$d n s$		$l = \frac{s}{n} + \frac{(n-1)d}{2}.$
5	$a d n$	s	$s = \frac{1}{2}n [2a + (n-1)d].$
6	$a d l$		$s = \frac{l+a}{2} + \frac{l^2-a^2}{2d}.$
7	$a n l$		$s = (l+a) \frac{n}{2}.$
8	$d n l$		$s = \frac{1}{2}n [2l - (n-1)d].$
9	$d n l$	a	$a = l - (n-1)d.$
10	$d n s$		$a = \frac{s}{n} - \frac{(n-1)d}{2}.$
11	$d l s$		$a = \frac{1}{2}d \pm \sqrt{(l + \frac{1}{2}d)^2 - 2ds}.$
12	$n l s$		$a = \frac{2s}{n} - l.$
13	$a n l$	d	$d = \frac{l-a}{n-1}.$
14	$a n s$		$d = \frac{2(s-an)}{n(n-1)}.$
15	$a l s$		$d = \frac{l^2-a^2}{2s-l-a}.$
16	$n l s$		$d = \frac{2(nl-s)}{n(n-1)}.$
17	$a d l$	n	$n = \frac{l-a}{d} + 1.$
18	$a d s$		$n = \frac{d-2a \pm \sqrt{(2a-d)^2 + 8ds}}{2d}.$
19	$a l s$		$n = \frac{2s}{l+a}.$
20	$d l s$		$n = \frac{2l+d \pm \sqrt{(2l+d)^2 - 8ds}}{2d}.$

EXERCISE 116.

1. Find the thirteenth term of 5, 9, 13.....
 ninth term of $-3, -1, 1$
 tenth term of $-2, -5, -8$
 eighth term of $a, a + 3b, a + 6b$
 fifteenth term of $1, \frac{1}{2}, \frac{1}{3}$
 thirteenth term of $-48, -44, -40$
2. The first term of an arithmetical series is 3, the thirteenth term is 55. Find the common difference.
3. Find the arithmetical mean between: (a.) 3 and 12;
 (b.) -5 and 17 ; (c.) $a^2 + ab - b^2$ and $a^2 - ab + b^2$.
4. Insert three arithmetical means between 1 and 19;
 and four arithmetical means between -4 and 17 .
5. The first term of a series is 2, and the common difference $\frac{1}{3}$. What term will be 10?
6. The seventh term of a series, whose common difference is 3, is 11. Find the first term.
7. Find the sum of
 $5 + 8 + 11 + \dots$ to ten terms.
 $-4 - 1 + 2 + \dots$ to seven terms.
 $a + 4a + 7a + \dots$ to n terms.
 $\frac{1}{3} + \frac{1}{15} + \frac{1}{15} + \dots$ to twenty-one terms.
 $1 + 2\frac{2}{3} + 4\frac{1}{3} + \dots$ to twenty terms.
8. The sum of six numbers of an arithmetical series is 27, and the first term is 1. Determine the series.
9. How many terms of the series $-5 - 2 + 1 + \dots$ must be taken so that their sum may be 63?
10. The first term is 12, and the sum of ten terms is 10. Find the last term.

11. The arithmetical mean between two numbers is 10, and the mean between the double of the first and the triple of the second is 27. Find the numbers.
12. Find the middle term of eleven terms whose sum is 66.
13. The first term of an arithmetical series is 2, the common difference is 7, and the last term 79. Find the number of terms.
14. The sum of 15 terms of an arithmetical series is 600, and the common difference is 5. Find the first term.
15. Insert ten arithmetical means between -7 and 114 .
16. The sum of three numbers in arithmetical progression is 15, and the sum of their squares is 83. Find the numbers.

Let $x - y$, x , $x + y$ represent the numbers.

17. Arithmetical means are inserted between 5 and 23, so that the sum of the first two is to the sum of the last two as 2 is to 5. How many means are inserted?
18. Find three numbers of an arithmetical series whose sum shall be 21, and the sum of the first and second shall be $\frac{2}{3}$ of the sum of the second and third.
19. Find three numbers whose common difference is 1, such that the product of the second and third exceeds the product of the first and second by $\frac{1}{2}$.
20. How many terms of the series 1, 4, 7,.... must be taken, in order that the sum of the first half may bear to the sum of the second half the ratio 10 : 31?
21. A travels uniformly 20 miles a day; B starts three days later, and travels 8 miles the first day, 12 the second, and so on, in arithmetical progression. In how many days will B overtake A?

22. A number consists of three digits which are in arithmetical progression; and this number divided by the sum of its digits is equal to 26; but if 198 be added to it, the digits in the units' and hundreds' places will be interchanged. Find the number.
23. The sum of the squares of the extremes of four numbers in arithmetical progression is 200, and the sum of the squares of the means is 136. What are the numbers?
24. Show that if any even number of terms of the series 1, 3, 5..... is taken, the sum of the first half is to the sum of the second half in the ratio 1 : 3.
25. A and B set out at the same time to meet each other from two places 343 miles apart. Their daily journeys were in arithmetical progression, A's increase being 2 miles each day, and B's decrease being 5 miles each day. On the day at the end of which they met, each travelled exactly 20 miles. Find the duration of the journey.
26. Suppose that a body falls through a space of $16\frac{1}{2}$ feet in the first second of its fall, and in each succeeding second $32\frac{1}{2}$ more than in the next preceding one. How far will a body fall in 20 seconds?
27. The sum of five numbers in arithmetical progression is 45, and the product of the first and fifth is $\frac{5}{8}$ of the product of the second and fourth. Find the numbers.
28. If a full car descending an incline draw up an empty one at the rate of $1\frac{1}{2}$ feet the first second, $4\frac{1}{2}$ feet the next second, $7\frac{1}{2}$ feet the third, and so on, how long will it take to descend an incline 150 feet in length? What part of the distance will the car have descended in the first half of the time?

GEOMETRICAL SERIES.

384. A series is called a **Geometrical Series** or a **Geometrical Progression** when each succeeding term is obtained by multiplying the preceding term by a *constant multiplier*.

385. The general representative of such a series will be

$$a, ar, ar^2, ar^3, ar^4, \dots,$$

in which a is the first term and r the constant multiplier or ratio.

386. Since the exponent of r increases by 1 for every term, the exponent will always be 1 less than the number of the term; so that the

$$nth \text{ term} = ar^{n-1}.$$

387. If the nth term is denoted by l , this equation becomes

$$l = ar^{n-1}. \quad (1)$$

388. The *geometrical mean* between two numbers is the number which stands between them, and makes with them a geometrical series.

389. If a and b denote two numbers, and G their geometrical mean, then, by definition of a geometrical series,

$$\begin{aligned} \frac{G}{a} &= \frac{b}{G}, \\ \therefore G &= \sqrt{ab}. \end{aligned} \quad (2)$$

390. Sometimes it is required to insert several geometrical means between two numbers.

If m = the number of means, then $m + 2 = n$, the whole number of terms; and if $m + 2$ is substituted for n in the equation

$$l = ar^{n-1},$$

the result is

$$l = ar^{m+1}.$$

$$\therefore r^{m+1} = \frac{l}{a}. \quad (3)$$

Thus, if it is required to insert three geometrical means between 3 and 48, the value of r is found to be

$$r^4 = \frac{48}{3} = 16.$$

$$\therefore r = 2,$$

and the series will be 3, 6, 12, 24, 48.

391. If l denotes the last term, a the first term, n the number of terms, r the common ratio, and s the sum of the n terms, then

$$s = a + ar + ar^2 + ar^3 + \dots + ar^{n-1}.$$

Multiply by r , $rs = ar + ar^2 + ar^3 + \dots + ar^{n-1} + ar^n$.

Therefore, by subtracting the first equation from the second,

$$rs - s = ar^n - a,$$

or

$$(r - 1)s = a(r^n - 1).$$

$$\therefore s = \frac{a(r^n - 1)}{r - 1}. \quad (4)$$

392. When r is less than 1, this formula will be more convenient if written

$$s = \frac{a(1 - r^n)}{1 - r}.$$

393. Since

$$l = ar^{n-1},$$

then

$$rl = ar^n,$$

and (4) may be written

$$s = \frac{rl - a}{r - 1}.$$

In working out the following results, the student will make use of the two equations, $l = ar^{n-1}$ and $s = \frac{a(r^n - 1)}{r - 1}$.

NO.	GIVEN.	REQUIRED.	RESULTS.
1	$a r n$	l	$l = ar^{n-1}.$
2	$a r s$		$l = \frac{a + (r-1)s}{r}.$
3	$a n s$		$l(s-l)^{n-1} - a(s-a)^{n-1} = 0.$
4	$r n s$		$l = \frac{(r-1)sr^{n-1}}{r^n - 1}.$
5	$a r n$	s	$s = \frac{a(r^n - 1)}{r - 1}.$
6	$a r l$		$s = \frac{rl - a}{r - 1}.$
7	$a n l$		$s = \frac{\sqrt[n-1]{ln} - \sqrt[n-1]{a^n}}{\sqrt[n-1]{l} - \sqrt[n-1]{a}}.$
8	$r n l$		$s = \frac{lr^n - l}{r^n - r^{n-1}}.$
9	$r n l$	a	$a = \frac{l}{r^{n-1}}.$
10	$r n s$		$a = \frac{(r-1)s}{r^n - 1}.$
11	$r l s$		$a = rl - (r-1)s.$
12	$n l s$		$a(s-a)^{n-1} - l(s-l)^{n-1} = 0.$
13	$a n l$	r	$r = \sqrt[n-1]{\frac{l}{a}}.$
14	$a n s$		$r^n - \frac{s}{a}r + \frac{s-a}{a} = 0.$
15	$a l s$		$r = \frac{s-a}{s-l}.$
16	$n l s$		$r^n - \frac{s}{s-l}r^{n-1} + \frac{l}{s-l} = 0.$
17	$a r l$	n	$n = \frac{\log l - \log a}{\log r} + 1.$
18	$a r s$		$n = \frac{\log [a + (r-1)s] - \log a}{\log r}.$
19	$a l s$		$n = \frac{\log l - \log a}{\log(s-a) - \log(s-l)} + 1.$
20	$r l s$		$n = \frac{\log l - \log [lr - (r-1)s]}{\log r} + 1.$

EXERCISE 117.

1. Find the seventh term of 2, 6, 18.....
 sixth term of 3, 6, 12.....
 ninth term of 6, 3, $1\frac{1}{2}$
 eighth term of 1, -2, 4.....
 twelfth term of x^3 , x^4 , x^5
 fifth term of $4a$, $-6ma^2$, $9m^2a^3$
2. Find the geometrical mean between $18x^3y$ and $30xy^2z$.
3. Find the ratio when the first and third terms are 5 and 80 respectively.
4. Insert two geometrical means between 8 and 125; and three between 14 and 224.
5. If $a=2$ and $r=3$, which term will be equal to 162?
6. The fifth term of a geometrical series is 48, and the ratio 2. Find the first and seventh terms.
7. Find the sum of

$3 + 6 + 12 + \dots$	to eight terms.
$1 - 3 + 9 - \dots$	to seven terms.
$8 + 4 + 2 + \dots$	to ten terms.
$0.1 + 0.5 + 2.5 + \dots$	to seven terms.
$m - \frac{m}{4} + \frac{m}{16} - \dots$	to five terms.
8. The population of a city increases in four years from 10,000 to 14,641. What is the rate of increase?
9. The sum of four numbers in geometrical progression is 200, and the first term is 5. Find the ratio.
10. Find the sum of eight terms of a series whose last term is 1, and fifth term $\frac{1}{8}$.

11. In an odd number of terms, show that the product of the first term and last term will be equal to the square of the middle term.
12. The product of four terms of a geometrical series is 4, and the fourth term is 4. Determine the series.
13. If from a line one-third be cut off, then one-third of the remainder, and so on, what fraction of the whole will remain when this has been done five times?
14. Of three numbers in geometrical progression, the sum of the first and second exceeds the third by 3, and the sum of the first and third exceeds the second by 21. What are the numbers?
15. Find two numbers whose sum is $3\frac{1}{2}$ and geometrical mean $1\frac{1}{2}$.
16. A glass of wine is taken from a decanter that holds ten glasses, and a glass of water poured in. After this is done five times, what part of the contents is wine?
17. There are four numbers such that the sum of the first and the last is 11, and the sum of the others is 10. The first three of these four numbers are in arithmetical progression, and the last three are in geometrical progression. Find the numbers.
18. Find three numbers in geometrical progression such that their sum is 13 and the sum of their squares is 91.
19. The difference between two numbers is 48, and the arithmetical mean exceeds the geometrical mean by 18. Find the numbers.
20. There are four numbers in geometrical progression, the second of which is less than the fourth by 24, and the sum of the extremes is to the sum of the means as 7 to 3. Find the numbers.

21. A number consists of three digits in geometrical progression. The sum of the digits is 13; and if 792 be added to the number, the digits in the units' and hundreds' places will be interchanged. Find the number.

394. When r is less than 1, a geometrical series has its terms continually decreasing; and by increasing n , the value of the n th term, ar^{n-1} , may be made as small as we please, though not absolutely zero.

395. The formula for the sum of n terms,

$$\frac{a(1-r^n)}{1-r}$$

may be written $\frac{a}{1-r} - \frac{ar^n}{1-r}$.

By increasing n indefinitely, the value of $\frac{ar^n}{1-r}$ becomes indefinitely small, so that the sum of n terms approaches indefinitely to $\frac{a}{1-r}$ as its limit.

Ex. Find the limit of $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \dots$

Here $a = 1$, and $r = -\frac{1}{2}$,

and therefore the limit $\frac{a}{1-r} = \frac{1}{1-(-\frac{1}{2})} = \frac{1}{1+\frac{1}{2}} = \frac{2}{3}$. Ans.

22. Find the limits of the sums of the following infinite series :

$4 + 2 + 1 + \dots$	$2 - 1\frac{1}{3} + \frac{8}{9} - \dots$
$\frac{1}{2} + \frac{1}{8} + \frac{1}{27} + \dots$	$0.1 + 0.01 + 0.001 + \dots$
$\frac{1}{4} - \frac{1}{18} + \frac{1}{81} - \dots$	$0.868686\dots$
$1 - \frac{2}{3} + \frac{4}{27} - \dots$	$0.54444\dots$
$\frac{1}{8} + \frac{1}{125} + \frac{1}{432} + \dots$	$0.83636\dots$

* HARMONICAL SERIES.

396. A series is called a **Harmonical Series**, or a **Harmonical Progression**, when the *reciprocals* of its terms form an *arithmetical series*.

Hence, the general representative of such a series will be

$$\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \dots, \frac{1}{a+(n-1)d}.$$

397. Questions relating to harmonical series should be solved by writing the reciprocals of its terms so as to form an arithmetical series.

398. If a and b denote two numbers, and H their harmonical mean, then, by the definition of a harmonical series,

$$\begin{aligned} \frac{1}{H} - \frac{1}{a} &= \frac{1}{b} - \frac{1}{H} \\ \therefore \frac{2}{H} &= \frac{1}{a} + \frac{1}{b} = \frac{a+b}{ab}; \\ \therefore H &= \frac{2ab}{a+b}. \end{aligned}$$

399. Sometimes it is required to insert several harmonical means between two numbers.

Ex. Let it be required to insert three harmonical means between 3 and 18.

Find the three arithmetical means between $\frac{1}{3}$ and $\frac{1}{18}$.

These are found to be $\frac{1}{9}$, $\frac{1}{6}$, $\frac{1}{4}$; therefore, the harmonical means are $\frac{3}{2}$, $\frac{7}{2}$, $\frac{9}{2}$; or $3\frac{1}{2}$, $5\frac{1}{2}$, 8 .

* A harmonical series is so called because musical strings of uniform thickness and tension produce *harmony* when their lengths are represented by the *reciprocals* of the natural series of numbers; that is, by the series, $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$, etc.

EXERCISE 118.

1. Insert four harmonical means between 2 and 12.
2. Find two numbers whose difference is 8 and the harmonical mean between them $1\frac{1}{2}$.
3. Find the seventh term of the harmonical series 3, $3\frac{1}{2}$, 4.....
4. Continue to two terms each way the harmonical series two consecutive terms of which are 15, 16.
5. The first two terms of a harmonical series are 5 and 6. Which term will equal 30?
6. The fifth and ninth terms of a harmonical series are 8 and 12. Find the first four terms.
7. The difference between the arithmetical and harmonical means between two numbers is $1\frac{1}{2}$, and one of the numbers is four times the other. Find the numbers.
8. Find the arithmetical, geometrical, and harmonical means between two numbers a and b ; and show that the geometrical mean is a mean proportional between the arithmetical and harmonical means. Also, arrange these means in order of magnitude.
9. The arithmetical mean between two numbers exceeds the geometrical by 13, and the geometrical exceeds the harmonical by 12. What are the numbers?
10. The sum of three terms of a harmonical series is 11, and the sum of their squares is 49. Find the numbers.
11. When a, b, c are in harmonical progression, show that $a:c::a-b:b-a$.

CHAPTER XXII.

* CHOICE. BINOMIAL THEOREM.

400. If three paths, *A*, *B*, and *C*, lead to the top of a mountain, there is obviously a choice of three different ways of ascending the mountain; and when the top of the mountain is reached, there is again a choice of three different ways of descending.

How many different ways are there of doing both?

If a traveller ascend by *A*, he may descend by *A*, *B*, or *C*. This makes three ways of doing both. If he ascend by *B*, he may descend by *A*, *B*, or *C*; and again, if he ascend by *C*, he may descend by *A*, *B*, or *C*.

Therefore, there are $3 \times 3 = 9$ ways in all of doing both.

These ways may be indicated as follows :

- | | | |
|----------------------------|----------------------------|----------------------------|
| 1. <i>A</i> and <i>A</i> . | 4. <i>B</i> and <i>A</i> . | 7. <i>C</i> and <i>A</i> . |
| 2. <i>A</i> and <i>B</i> . | 5. <i>B</i> and <i>B</i> . | 8. <i>C</i> and <i>B</i> . |
| 3. <i>A</i> and <i>C</i> . | 6. <i>B</i> and <i>C</i> . | 9. <i>C</i> and <i>C</i> . |

401. Suppose the traveller does not wish to ascend and descend by the *same* path, then what choice has he?

He has a choice of three different ways in ascending.

But he has a choice of only *two* ways in descending.

If each of the three ways of ascending be joined to the two eligible ways of descending, the result is $3 \times 2 = 6$ ways of doing both.

* This chapter is based upon Whitworth's *Choice and Chance*, and many of the examples have been taken from that elegant work.

These ways are :

- | | | |
|----------------------------|----------------------------|----------------------------|
| 1. <i>A</i> and <i>B</i> . | 3. <i>B</i> and <i>A</i> . | 5. <i>C</i> and <i>A</i> . |
| 2. <i>A</i> and <i>C</i> . | 4. <i>B</i> and <i>C</i> . | 6. <i>C</i> and <i>B</i> . |

402. If a box contains five capital letters, *A, B, C, D, E*, and three small letters, *x, y, z*, in how many different ways may a capital letter and a small letter be selected ?

A capital letter may be selected in 5 ways.

With each capital letter selected, a small letter can be joined in 3 ways. So that the number of different ways in which the selection can be made is $3 \times 5 = 15$. These ways are :

<i>Ax</i>	<i>Bx</i>	<i>Cx</i>	<i>Dx</i>	<i>Ex</i>
<i>Ay</i>	<i>By</i>	<i>Cy</i>	<i>Dy</i>	<i>Ey</i>
<i>Az</i>	<i>Bz</i>	<i>Cz</i>	<i>Dz</i>	<i>Ez</i>

403. Hence the fundamental principle of choice :

I. If one thing can be done in a different ways, and (when it has been done in any one of these ways) another thing can be done in b different ways, then both can be done in $a \times b$ different ways.

For, corresponding to *each* of the *a* ways of doing the first thing, there are *b* ways of doing the second thing. Therefore, altogether, there are $a \times b$ ways of doing both things.

(1) On a shelf are 7 English and 5 French books. In how many ways can one of each be chosen ?

$7 \times 5 = 35$. *Ans.*

(2) On a shelf are 7 English, 5 French, and 9 German books. In how many ways can two books be chosen so that they shall be in different languages ?

An English book and a French book can be chosen in $7 \times 5 = 35$ ways. A French book and a German book in $5 \times 9 = 45$ ways. An English book and a German book in $7 \times 9 = 63$ ways.

Hence, there is a choice of $35 + 45 + 63 = 143$ ways. *Ans.*

- (3) Out of 8 different pairs of gloves, in how many different ways can a right-hand and a left-hand glove be chosen which shall not form a pair?

A right-hand glove can be chosen in 8 ways; and when it is chosen there are 7 left-hand gloves, any one of which may be put with it without making a pair. Hence, the choice is in $8 \times 7 = 56$ ways.

- (4) Out of the ten digits, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, how many numbers each consisting of two figures can be formed?

Since 0 has no value in the left-hand place, the left-hand place can be filled in 9 ways.

The right-hand place can be filled in 10 ways, since repetitions of the digits are allowed (as 22, 33, etc.).

Hence, the whole number is $9 \times 10 = 90$.

- (5) How many odd numbers consisting of two figures can be formed with the ten digits?

The left-hand place can be filled in 9 ways; the right-hand place in only 5 ways, since it must be either 1, 3, 5, 7, or 9.

Hence, the number is, $9 \times 5 = 45$. *Ans.*

404. By a simple extension of Rule I. it is evident that,

II. If one thing can be done in a ways, and then a second thing can be done in b ways, then a third in c ways, then a fourth in d ways, etc., the number of ways of doing all the things will be $a \times b \times c \times d$, etc.

- (1) In how many ways can four Christmas presents be given to four boys, one to each boy?

The first present may be given to any one of the boys; hence there are 4 ways of disposing of it.

The second present may be given to any one of the other three boys; hence there are 3 ways of disposing of it.

The third present may be given to either of the other two boys; hence there are 2 ways of disposing of it.

The fourth present must be given to the last boy ; hence there is only one way of disposing of it.

There are, then, $4 \times 3 \times 2 \times 1 = 24$ ways.

- (2) In how many ways can five presents be given to two children ?

Each present may be disposed of in 2 ways ; for it may be given to either child. Hence, the whole number of ways of giving the presents is $2 \times 2 \times 2 \times 2 \times 2 = 32$.

- (3) In how many ways can five presents be *divided* between two children ?

This question differs from the last only in the fact that a *division* of the gifts excludes the two ways in which either child receives *all* the gifts.

Hence, there are $32 - 2 = 30$ ways.

- (4) In how many ways can x things be given to n persons ?
 n^x . *Ans.*

- (5) In how many ways can a vowel and a consonant be chosen out of the alphabet ?

Since there are in the alphabet 6 vowels and 20 consonants, a vowel can be chosen in 6 ways and a consonant in 20 ways, and both (Rule I.) in $6 \times 20 = 120$ ways.

- (6) In how many ways can a two-lettered word be made, containing one vowel and one consonant ?

The vowel can be chosen in 6 ways and the consonant in 20 ways ; and then each combination of a vowel and a consonant can be written in 2 ways ; as *ac*, *ca*.

Hence, the whole number of ways is $6 \times 20 \times 2 = 240$.

405. The last two examples show the difference between a **selection** or **combination** of different things, and an **arrangement** or **permutation** of the same things.

Thus, *ac* forms a *combination* of a vowel and a consonant, and *ac* and *ca* form two different arrangements of this combination.

From (5) it is seen that 120 different *combinations* can be made with a vowel and a consonant; and from (6) it is seen that 240 different *permutations* can be made with the same.

Again, *a, b, c* is a selection of three letters from the alphabet. This selection then admits of 6 different arrangements or permutations, as follows:

<i>abc</i>	<i>bca</i>	<i>cab</i>
<i>acb</i>	<i>bac</i>	<i>cba</i>

406. A *selection* or *combination* of any number of elements or things means a group of that number of elements or things put together without regard to their order of sequence. An *arrangement* or *permutation* of any number of elements or things means a group of that number of elements or things put together with reference to their order of sequence.

ARRANGEMENTS OR PERMUTATIONS.

407. In how many ways can the letters of the word Cambridge be arranged, taken all at a time?

There are nine letters. In making any arrangement any one of the letters may be put in the first place. Hence, the first place can be filled in 9 ways. Then the second place can be filled with any one of the remaining eight letters; that is, in 8 ways.

In like manner, the third place in 7 ways, the fourth place in 6 ways, and so on; and, lastly the ninth place in 1 way.

If the nine places are indicated by Roman numerals, the result (Rule II.) is as follows:

I. II. III. IV. V. VI. VII. VIII. IX.

$$9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 362,880 \text{ ways.}$$

408. Hence, it will be seen that,

III. *The number of arrangements or permutations of n different elements or things taken all at a time is*

$$n (n - 1) (n - 2) (n - 3) \dots 3 \times 2 \times 1.$$

For, the first place may be filled in n ways, then the second place in $n - 1$ ways, then the third place in $n - 2$ ways, and so on to the last place, which can be filled in only 1 way.

Hence (Rule II.) the whole number of arrangements is the continued product of all these numbers,

$$n(n-1)(n-2)(n-3) \dots 3 \times 2 \times 1.$$

409. For the sake of brevity this product is written $\lfloor n$, and is read **factorial n** .

410. It will also be evident that,

IV. *The number of arrangements of n different elements, taken r at a time, is*

$$n(n-1)(n-2) \dots \text{to } r \text{ factors,}$$

$$\text{that is, } n(n-1)(n-2) \dots [n-(r-1)],$$

$$\text{or } n(n-1)(n-2) \dots (n-r+1).$$

For the first place can be filled in n ways, the second in $n - 1$ ways, the third place in $n - 2$ ways, and the r th place in $n - (r - 1)$ ways.

- (1) A shelf contains 4 English books, 5 French books, and 6 German books; in how many ways can these books be arranged?

$$\lfloor 15 = 1,307,674,368,000 \text{ ways.}$$

- (2) In how many ways can these books be arranged, if the books of each language are kept together?

The English books can be arranged (Rule III.) in $\lfloor 4$ ways, the French books in $\lfloor 5$ ways, and the German books in $\lfloor 6$ ways. Also, the three sets can be arranged in $\lfloor 3$ different orders. Hence, the number of ways in which the whole can be done is (Rule II.)

$$\lfloor 4 \times \lfloor 5 \times \lfloor 6 \times \lfloor 3 = 12,441,600.$$

(3) Of the arrangements possible with the letters of the word *Cambridge*:

- (i.) How many will begin with a vowel?
- (ii.) How many will both begin and end with a vowel?

In filling the nine places of any arrangement in case (i.), the first place can be filled in only 3 ways, the other places in $\underline{8}$ ways.

In case (ii.) the first place can be filled in 3 ways, the last place in 2 ways (one vowel having been used), and the remaining seven places in $\underline{7}$ ways.

Hence, the answer to (i.) is $3 \times \underline{8} = 120,960$;
to (ii.) is $3 \times 2 \times \underline{7} = 30,240$.

(4) With the letters of the word *Cambridge*, how many arrangements can be made:

- (i.) Each beginning with the word *Cam*?
- (ii.) Keeping the letters *Cam* always together?

For case (i.) the answer is evidently $\underline{6}$; since our only choice lies in arranging the remaining six letters of the word.

Case (ii.) may be resolved into arranging *Cam* and the last six letters, regarded as seven distinct elements, and then arranging the letters *Cam*.

The first can be done in $\underline{7}$ ways, and the second in $\underline{3}$ ways. Hence (Rule II.) both can be done in $\underline{7} \times \underline{3} = 30,240$ ways.

(5) In how many ways can the letters of the word *Cambridge* be written:

- (i.) Without changing the *place* of any vowel?
- (ii.) Without changing the *order* of any vowel?
- (iii.) Without changing the *relative order* of vowels and consonants?

In (i.) the second, sixth, and ninth places can be filled each in only 1 way; the other places in $\underline{6}$ ways.

Therefore, the whole number of ways is $\underline{6} = 720$.

In (ii.) the vowels in the different arrangements are always kept in the order *a, i, e*. One of the six consonants can be placed in 4 ways: *before a, between a and i, between i and e, and after e*.

Then a second consonant can be placed in 5 ways, a third consonant in 6 ways, a fourth consonant in 7 ways, a fifth consonant in 8 ways, and the last consonant in 9 ways. Hence (Rule II.) the whole number of ways is $4 \times 5 \times 6 \times 7 \times 8 \times 9 = 60,480$.

In (iii.) the vowels can be arranged in $\underline{3}$ ways, and the consonants in $\underline{6}$ ways. Hence (Rule II.) the number of ways is $\underline{3} \times \underline{6} = 4320$

- (6) In how many ways can 4 persons, *A, B, C, D*, sit at a round table?

If the four places are not regarded as relative to each other, then the whole number of ways is $\underline{4} = 24$. But if the four places are regarded as relative to each other, then by placing one as *A*, in one position, and by arranging the others in the other three positions, the whole number of ways is $\underline{3} = 6$.

- (7) In how many ways can 6 persons form a ring?

Here relative position is required. Hence, the whole number of ways is $\underline{5} = 120$.

- (8) How many three-lettered words can be made from the alphabet, no letter being repeated in the same word?

$$26 \times 25 \times 24 = 15,600. \text{ Ans.}$$

- (9) How many four-lettered words?

$$26 \times 25 \times 24 \times 23 = 358,800. \text{ Ans.}$$

- (10) How many different arrangements can be made of the letters in the word *eye*?

Distinguish the *e*'s thus, e_1, e_2 , and arrange as follows:

$$\begin{array}{lll} e_1 y e_2 & e_1 e_2 y & y e_1 e_2 \\ e_2 y e_1 & e_2 e_1 y & y e_2 e_1 \end{array}$$

These six arrangements become three when the *e*'s are not distinguished. That is, each pair of arrangements produced by permuting the *e*'s is reduced to a single arrangement.

Hence, the number of different arrangements is found by dividing $\underline{3}$ by $\underline{2}$; that is, by dividing the number of arrangements possible when the letters are all different by the number of ways in which the two *e*'s can be permuted.

411. In how many different orders can the letters a, a, a, x, y be written ?

If the letters were all different, the answer (Rule III.) would be $\underline{5}$; but the three a 's may be permuted in $\underline{3} = 6$ ways.

Hence, the $\underline{5}$ arrangements may be divided into six groups, each group constituting but a single arrangement of the given letters.

Hence, the whole number of given orders is $\frac{\underline{5}}{\underline{3}} = 20$.

By the same course of reasoning, the whole number of different orders of a, a, a, x, x is found to be

$$\frac{\underline{5}}{\underline{3}\underline{2}} = 10.$$

412. In like manner for any other numbers. Hence,

V. *The number of arrangements of n elements, of which p are alike, q others are alike, and r others are alike, is*

$$\frac{\underline{n}}{\underline{p}\underline{q}\underline{r}\dots}.$$

- (1) In how many ways can the letters of the word *Mississippi* be arranged ?

$$\frac{\underline{11}}{\underline{4}\underline{4}\underline{2}} = 34,560. \text{ Ans.}$$

- (2) In how many different orders can a row of 4 white balls and 3 black balls be arranged ?

$$\frac{\underline{7}}{\underline{4}\underline{3}} = 35. \text{ Ans.}$$

- (3) In how many ways can 4 white balls and 3 black balls be placed in a row, if the balls are all different in size ?

$$\underline{7} = 5040. \text{ Ans.}$$

413. In case the n elements to be arranged are all different, *but repetitions of them are allowed*, then in making any arrangement, every place to be filled can be filled in n ways, since we may always repeat an element already used. Hence, corresponding to Rules III. and IV., the following rules apply to cases in which repetitions are allowed:

VI. *The number of arrangements of n different elements, taken all at a time, when repetitions are allowed, is n^n .*

VII. *The number of arrangements of n different elements, taken r at a time, when repetitions are allowed, is n^r .*

- (1) How many three-lettered words can be made from the alphabet, when repetitions are allowed?

$26^3 = 17,576$. *Ans.*

- (2) How many three-lettered words can be made from the 6 vowels when repetitions are allowed?

$6^3 = 216$. *Ans.*

- (3) A railway signal has 3 arms, and each arm may take four different positions, including the position of rest. How many signals in all can be made?

$4^3 - 1 = 63$. *Ans.*

- (4) In the common system of notation, how many numbers can be formed consisting of not more than 5 digits?

All the possible numbers may be regarded as consisting of each 5 digits, by prefixing zeros to the numbers consisting of less than 5 digits. Thus, 247 may be written 00247.

Hence, every possible arrangement of 5 digits out of the 10 digits will give one of the required numbers except 00000; and the answer is $10^5 - 1 = 99999$; that is, all the numbers between 0 and 100,000.

- (5) With the digits 0, 1, 2, 3, 4, 5, how many numbers between 1000 and 4000 can be formed?

Here we have to fill, in each possible case, four places. The first place can be filled with 1, 2, or 3, that is, in 3 ways; the second, third, and fourth places each in 6 ways. Therefore, the answer is $3 \times 6^3 = 648$.

- (6) With the same digits, 0, 1, 2, 3, 4, 5, and between the same limits, 1000 and 4000 :
- (i.) How many *even* numbers can be formed?
 - (ii.) How many *odd* numbers can be formed?
 - (iii.) How many numbers divisible by 5?

Evidently (i.) and (ii.) are like Ex. (5), except that the last place can be filled in only 3 ways. In (i.) the last place must be filled by 0, 2, or 4; in (ii.) the last place must be filled by 1, 3, or 5.

Hence, the answer in each of these cases is $3 \times 6 \times 6 \times 3 = 324$.

In (iii.) the last digit must be either 0 or 5; and the answer for this case is $3 \times 6 \times 6 \times 2 = 216$.

COMBINATIONS.

- 414.** In how many ways can 3 vowels be selected from the 5 vowels *a, e, i, o, u*?

The number of ways in which we can *arrange* 3 vowels out of 5 is (Rule IV.) $5 \times 4 \times 3 = 60$.

These 60 *arrangements* might be obtained by first forming all the possible *selections* of the 3 vowels out of 5, and then arranging the 3 vowels in each selection in as many ways as possible.

The 3 vowels of each selection may be arranged in $\underline{3} = 6$ ways.

Hence (Rule II.),

The number of selections $\times 6 =$ the number of arrangements $= 60$.

Therefore, the number of selections $= \frac{60}{6} = 10$.

- 415.** In general,

VIII. Out of *n* different elements, the number of selections of *r* elements is equal to the number of arrangements of *n* elements divided by \underline{r} .

For, let s = the number of ways of selecting r elements out of n elements. Then the r elements thus selected may be arranged (Rule III.) in \underline{r} different ways. Therefore (Rule I.) $s \times \underline{r}$ = the number of arrangements of n elements taken r at a time.

$$\therefore s = \frac{\text{the number of arrangements}}{\underline{r}}.$$

(By Rule IV.) The numerator of this fraction is equal to

$$\begin{aligned} & n(n-1)(n-2) \dots [n-(r-1)]. \\ \therefore s = & \frac{n(n-1)(n-2) \dots (n-r+1)}{\underline{r}}. \end{aligned}$$

If both terms of this fraction are multiplied by $\underline{n-r}$,

the result is
$$s = \frac{\underline{n}}{\underline{r} \underline{n-r}}.$$

If $n-r = p$, then $n = p+r$, and this formula may be written

$$s = \frac{\underline{p+r}}{\underline{p} \underline{r}}.$$

416. The value of this fraction is not altered if p and r are interchanged. Hence,

IX. Out of $p+r$ different elements, the number of ways in which p elements can be selected is the same as the number of ways in which r elements can be selected.

Thus, out of 8 elements, 3 elements can be selected in the same number of ways as 5 elements; namely,

$$\frac{\underline{8}}{\underline{3} \underline{5}} = \frac{8 \cdot 7 \cdot 6}{\underline{3}} = 56 \text{ ways.}$$

(1) Out of 20 consonants, in how many ways can 18 be selected?

The 18 can be selected in the same number of ways as 2; and the number of ways in which 2 can be selected (Rule VIII.) is

$$\frac{20 \times 19}{2} = 190. \text{ Ans.}$$

- (2) In how many ways can the same choice be made so as always to include the letter B ?

Taking B first we must then select 17 out of the remaining 19 consonants. This can be done in

$$\frac{19 \times 18}{2} = 171 \text{ ways.}$$

- (3) In how many ways can the same choice be made so as to include B and not to include C ?

Taking B first, we have then to choose 17 out of 18, C being excluded. This can be done in 18 ways.

- (4) A society consists of 50 members, 10 of whom are physicians. In how many ways can a committee of 6 members be selected so as to include 1 physician?

The 5 members not physicians can be selected in

$$\frac{|40}{|5|35} \text{ ways,}$$

and the 1 physician in 10 ways. Hence, the 6 can be selected in

$$10 \times \frac{|40}{|5|35} \text{ different ways.}$$

- (5) In how many ways can a committee of 6 members be selected so as to include *at least* 1 physician?

Six members can be selected from the *whole* society in

$$\frac{|50}{|6|44} \text{ ways.}$$

Six members can be selected from the whole society, so as to include *no physician*, by choosing them all from the 40 members who are not physicians, and this can be done in

$$\frac{|40}{|6|34} \text{ ways.}$$

Hence, $\frac{|50}{|6|44} - \frac{|40}{|6|34} =$ the number of ways of selecting the committee so as to include at least one physician.

- (6) Out of 20 Republicans and 6 Democrats, what choice is there of appointing a committee consisting of 3 Republicans and 2 Democrats ?

The Republicans can be selected in $\frac{20 \times 19 \times 18}{1 \times 2 \times 3} = 1140$ ways ; and the Democrats in $\frac{6 \times 5}{1 \times 2} = 15$ ways. Hence, the whole committee can be appointed in $1140 \times 15 = 17,100$ ways.

- (7) From 20 Republicans and 6 Democrats, in how many ways may 5 different offices be filled, three of which must be filled by Republicans, and the other two by Democrats ?

The first three offices can be assigned to 3 Republicans in $20 \times 19 \times 18 = 6840$ ways (Rule II.) ; and the other two offices can be assigned to 2 Democrats in $6 \times 5 = 30$ ways.

There is, then, a choice of $6840 \times 30 = 205,200$ different ways.

- (8) Out of 20 consonants and 6 vowels, in how many ways can we make a word consisting of 3 different consonants and 2 different vowels ?

Three consonants can be selected in $\frac{20 \times 19 \times 18}{1 \times 2 \times 3} = 1140$ ways, and 2 vowels in $\frac{6 \times 5}{1 \times 2} = 15$ ways. Hence (Rule I.) the 5 letters can be selected in $1140 \times 15 = 17,100$ ways.

When they have been so selected, they can be arranged (Rule III.) in $5! = 120$ different orders. Hence, there are $17,100 \times 120 = 2,052,000$ different ways of making the word.

- (9) How many words of 2 consonants and 1 vowel can be formed from 6 consonants and 3 vowels, the vowel being the middle letter of each word ?

The two consonants can be selected in 15 ways ; the vowel in 3 ways. Each combination of the two consonants and 1 vowel can be arranged in $2 \times 1 \times 1 = 2$ ways. Hence, the number of words that can be formed is $15 \times 3 \times 2 = 90$.

- (10) How many words of 3 consonants and 3 vowels can be formed from the alphabet, if one of the vowels is to be always *a*?

The consonants can be selected in $\frac{20 \times 19 \times 18}{1 \times 2 \times 3} = 1140$ ways, and the vowels in $\frac{5 \times 4}{1 \times 2} = 10$ ways.

Then the six letters of each combination can be arranged in $6! = 720$ ways. Hence the number of words that can be formed is $1140 \times 10 \times 720 = 8,208,000$.

417. To find for what value of *r* the number of selections of *n* things, taken *r* at a time, is greatest.

$$\text{The formula } s = \frac{n(n-1)(n-2)\dots(n-r+1)}{1 \times 2 \times 3 \times \dots r}$$

$$\text{may be written } s = \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \dots \frac{n-r+1}{r}.$$

The numerators of the factors on the right side of this equation begin with *n*, and form a descending series with the common difference 1; and the denominators begin with 1, and form an ascending series with the common difference 1. Therefore, from some point in the series, these factors become less than 1. Hence, the maximum product is reached when that product includes *all* the factors *greater* than 1.

When *n* is an *odd* number, the numerator and the denominator of each factor will be alternately both odd and both even; so that the factor greater than 1, but nearest to 1, will be the factor whose numerator exceeds the denominator by 2. Hence, in this case, *r* must have such a value that

$$n - r + 1 = r + 2, \text{ or } r = \frac{n-1}{2}.$$

When *n* is an *even* number, the numerator of the first factor will be even and the denominator odd; the numerator of the second factor will be odd and the denominator even; and so on, alternately; so that the factor greater than 1, but nearest to 1, will be the factor whose numerator exceeds the denominator by 1. Hence, in this case, *r* must have such a value that

$$n - r + 1 = r + 1, \text{ or } r = \frac{n}{2}.$$

- (1) What value of r will give the greatest number of selections out of 7 things ?

Here n is odd, and $r = \frac{n-1}{2} = \frac{7-1}{2} = 3.$

$$\therefore s = \frac{7 \times 6 \times 5}{1 \times 2 \times 3} = 35. \text{ Ans.}$$

If $r = 4$, then $s = \frac{7 \times 6 \times 5 \times 4}{1 \times 2 \times 3 \times 4} = 35.$

So that, when the number of things is odd, there will be two equal numbers of selections ; namely, *when the number of things taken together is just under and just over one-half of the whole number.*

- (2) What value of r will give the greatest number of selections out of 8 things ?

Here n is even, and $r = \frac{n}{2} = \frac{8}{2} = 4.$

$$\therefore s = \frac{8 \times 7 \times 6 \times 5}{1 \times 2 \times 3 \times 4} = 70. \text{ Ans.}$$

So that, when the number of things is even, the number of selections will be greatest *when one-half of the whole are taken together.*

418. It may be shown that,

X. *The number of ways in which $x + y$ different elements can be divided into two classes, so that one shall contain x and the other y elements, is equal to the number of ways in which either x elements or y elements may be selected from the $x + y$ elements ; or*

$$\frac{|x+y|}{|x|y}.$$

For each division of $x + y$ elements into two classes, one consisting of x elements, the other of y elements, is evidently effected by making a selection of x elements from $x + y$ elements, and leaving y elements not selected.

In like manner the number of ways in which $x + y + z$ different elements can be divided into 3 classes, containing x , y , and z elements respectively, is

$$\frac{|x + y + z|}{|x| |y| |z|}; \quad \text{and so on.}$$

- (1) In how many ways can 18 men be divided into 2 classes of 6 and 12?

$$\frac{|18|}{|6| |12|}.$$

- (2) In how many ways can 18 men be divided into 2 groups of 9 each?

According to the rule, the answer would be

$$\frac{|18|}{|9| |9|}.$$

The two groups, considered as groups, have no distinction; therefore, permuting them gives no new arrangement, and the true result is obtained by dividing the preceding by $|2|$, and is

$$\frac{|18|}{|2| |9| |9|}.$$

If any condition is added that shall make the two groups *different*, as, if one group wear red badges and the other blue, then the answer will be

$$\frac{|18|}{|9| |9|}.$$

419. Whenever the groups are indifferent, Rule IX. gives each arrangement repeated as many times as the groups can be permuted one with another; that is, $|2|$ when there are 2 groups, $|3|$ when there are 3 groups, and so on.

Hence, the result found by the rule must be divided by $|2|$, $|3|$, etc., in order to obtain the true result.

- (1) In how many ways can the 52 cards in a pack be divided among 4 players, each to have 13?

Here the assignment of each group to a different player makes the groups *different*; and the answer is

$$\frac{|52|}{|13|13|13|13|}$$

- (2) In how many ways can the 52 cards of a pack be divided into 4 piles?

Here the groups are *indifferent*, and the answer is

$$\frac{|52|}{4|13|13|13|13|}$$

- (3) A boat's crew consists of 8 men, of whom 2 can row only on the stroke side of the boat, and 3 can row only on the bow side. In how many ways can the crew be arranged?

There are left 3 men who can row on either side; 2 of these must row on the stroke side, and one on the bow side.

The number of ways in which these three can be divided is

$$\frac{|3|}{|2|1|} = 3 \text{ ways.}$$

When the stroke side is completed, the 4 men can be arranged in $|4|$ ways; likewise, the 4 men of the bow side can be arranged in $|4|$ ways. Hence (Rule II.) the arrangement can be made in $3 \times |4|4 = 1728$ ways.

- (4) In how many ways can 10 copies of Homer, 6 of Virgil, and 4 of Horace be given to 20 boys, so that each boy may receive a book?

The boys have to form themselves into a group of 10 for Homer, of 6 for Virgil, of 4 for Horace. This can be done in

$$\frac{|20|}{|10|6|4|} \text{ different ways.}$$

- (5) In how many ways can 3 copies of one book, 2 of another, and 1 of a third be given to a class of 12 boys, so that no boy shall receive more than 1 book?

In every possible way of assigning the books, 6 boys receive them. These 6 may be selected from the whole 12 (Rule VIII.) in

$$\frac{\overline{12}}{\overline{6} \overline{6}} \text{ ways.}$$

When thus selected, the books may be assigned to them (Rule V.) in

$$\frac{\overline{6}}{\overline{3} \overline{2} \overline{1}} \text{ ways.}$$

Hence, the whole number of ways of giving the books is

$$\frac{\overline{12}}{\overline{6} \overline{6}} \times \frac{\overline{6}}{\overline{3} \overline{2} \overline{1}}.$$

- (6) In how many ways can the same books be given to the 12 boys, so that no boy shall receive more than 1 copy of any book?

In allotting the three copies of the first book, the boys are to be separated into two groups of 3 and 9; and the group of 3 will in each case receive the books.

This can be done in $\frac{\overline{12}}{\overline{3} \overline{9}}$ ways.

Likewise the two copies of the second book may be given in

$$\frac{\overline{12}}{\overline{2} \overline{10}} \text{ ways.}$$

The single copy of the third book can be given in 12 ways.

Hence (Rule II.) the books may be given in

$$\frac{\overline{12}}{\overline{3} \overline{9}} \times \frac{\overline{12}}{\overline{2} \overline{10}} \times 12 \text{ different ways.}$$

- (7) In how many ways can 2 letters be selected from a, b, c, d, e, f , if the letters may be *repeated* in making the selection?

Without repetitions, two letters can be selected from 6 in 15 ways. With repetitions, as aa , etc., 6 selections can be made. Hence, there are $15 + 6 = 21$ different selections.

- (8) In how many ways can selections of 3 letters be made from a, b, c, d, e, f , if letters may be repeated?

These selections will be of 3 kinds:

- (i.) All three letters different.
- (ii.) Two letters alike, the third different.
- (iii.) All three letters alike.

(By Rule VIII.) (i.) gives 20 ways.

(ii.) gives $6 \times 5 = 30$; for we can choose 2 alike in 6 ways, and then join a different letter to each pair in 5 ways.

(iii.) gives evidently 6 ways.

Hence, there are in all $20 + 30 + 6 = 56$ different selections.

420. This may also be shown as follows:

By adding the 6 letters, a, b, c, d, e, f , to each selection of the kind required, they will become selections of $6 + 3 = 9$ letters out of 6, in which selections each letter occurs *at least once*, and in which, therefore, there must be *exactly* 3 repetitions. These repetitions may be all of the same letter, or divided among the different letters.

Any one of these selections may be made by writing 9 places, and then filling them in alphabetical order, taking care to make exactly three repetitions in passing from one end of the row to the other.

Thus:

1	2	3	4	5	6	7	8	
a	a	a	a	b	c	d	e	f
a	b	c	c	c	d	e	e	f
a	b	c	c	d	d	e	e	f

By striking out the 6 letters, a, b, c, d, e, f , each once, there is left in the first row aaa , in the second row cee , in the third row cde ; that is, three of the required selections.

Now, in filling the 9 places for each row, $9 - 1$ or 8 steps must be made, of which exactly 3 are repetitions, and each of the other 5 is a change to a different letter.

The 3 repetitions may be chosen from the 8 steps (Rule VIII.) in

$$\frac{8 \times 7 \times 6}{1 \times 2 \times 3} = 56 \text{ ways.}$$

This, then, is the number of ways in which 3 letters can be selected out of 6 letters, when *repetitions* are allowed.

In other words, 3 elements can be selected from 6 elements, when repetitions are allowed, in as many ways as 3 elements can be selected from $6 + 3 - 1$, or 8 elements *without repetitions*.

421. Hence the general rule :

XI. *The number of ways of selecting r elements from n different elements, when repetitions are allowed, is the same as the number of ways of selecting r elements from $n + r - 1$ elements without repetitions.*

And this number of ways is (Rule VIII.).

$$\frac{n+r-1}{r} = \frac{n(n+1)(n+2) \dots (n+r-1)}{r}$$

- (1) In how many ways can 4 elements be selected from n elements, when repetitions are allowed ?

$$\frac{n(n+1)(n+2)(n+3)}{4} \text{ Ans.}$$

- (2) How many dominoes are there in a set numbered from double blank to double nine ?

Each domino is made by selecting two numbers out of the ten digits, and repetitions are included ; that is, the two numbers on a domino may be the same.

Hence, the number of dominoes is equal to the number of selections of 2 from $10 + 2 - 1$, or 11, without repetitions.

$$\frac{10 \times 11}{1 \times 2} = 55. \text{ Ans.}$$

- (3) In how many ways can 4 glasses be filled with 5 kinds of wine, without mixing?

The number is equal to the number of ways in which 4 things can be selected from $5 + 4 - 1$, or 8 things, without repetitions.

$$\frac{5 \times 6 \times 7 \times 8}{1 \times 2 \times 3 \times 4} = 70. \text{ Ans.}$$

- (4) In how many ways can six rugs be selected at a shop where 2 kinds of rugs are sold?

The number is equal to the number of ways of selecting 6 from $2 + 6 - 1 = 7$, without repetitions.

$$\frac{2 \times 3 \times 4 \times 5 \times 6 \times 7}{1 \times 2 \times 3 \times 4 \times 5 \times 6} = 7.$$

If a and b represent the two kinds of rugs, the 7 ways are as follows:

$a a a a a a$	$a a a b b b$
$a a a a a b$	$a a b b b b$
$a a a a b b$	$a b b b b b$
$b b b b b b$	

422. It may be shown that,

XII. *The number of ways in which a selection (of some, or all) can be made from n different things is $2^n - 1$.*

For each thing can be either taken or left, that is, can be disposed of in two ways.

There are n things; hence (Rule II.) they can all be disposed of in 2^n ways. But, among these ways is included the case in which all are rejected; and this case is inadmissible.

Hence, the number of ways of making the selection is $2^n - 1$.

- (1) In a shop window 20 different articles are exposed for sale. What choice has a purchaser?

$$2^{20} - 1 = 1,048,575. \text{ Ans.}$$

- (2) How many different amounts can be weighed with 1 lb., 2 lb., 4 lb., 8 lb., and 16 lb. weights?

$$2^5 - 1 = 31. \text{ Ans.}$$

(Let the student write out the 31 weights.)

423. It may be shown that,

XIII. *The whole number of ways in which a selection can be made from $p + q + r$ things, of which p are alike, q are alike, r are alike, etc., is $(p + 1)(q + 1)(r + 1)$ $- 1$.*

For the set of p things may be disposed of in $p + 1$ ways, since none of them may be taken, or 1, 2, 3,, or p , may be taken.

In like manner, the q things may be disposed of in $q + 1$ ways; the r things in $r + 1$; and so on.

Hence (Rule II.), all the things may be disposed of in $(p + 1)(q + 1)(r + 1)$ ways.

But the case in which *all* the things are rejected is inadmissible; hence, the whole number of ways is $(p + 1)(q + 1)(r + 1)$ $- 1$.

- (1) In how many ways can 2 boys divide between them 10 oranges all alike, 15 apples all alike, and 20 peaches all alike?

Here, the case in which the first boy takes none, and the case in which the second boy takes none, must be rejected.

Therefore, the answer is 1 less than the result, according to Rule XIII. $11 \times 16 \times 21 - 2 = 3694$. *Ans.*

- (2) If there are m kinds of things, and n things of each kind, in how many ways can a selection be made?

• In this case p, q, r , etc., are all equal, and each is equal to n . Hence the result is $(n + 1)^m - 1$.

- (3) If there are m kinds of things, and 1 thing of the first kind, 2 of the second, 3 of the third, and so on, in how many ways can a selection be made?

$m + 1$ $- 1$. *Ans.*

EXERCISE 119.

1. How many different permutations can be made of the letters in the word *Ecclesiastical*, taken all together?
2. Of all the numbers that can be formed with four of the digits 5, 6, 7, 8, 9, how many will begin with 56?

3. If the number of permutations of n things, taken 4 together, is equal to 12 times the permutations of n things, taken 2 together, find n .
4. With 3 consonants and 2 vowels, how many words of 3 letters can be formed, beginning and ending with a consonant, and having a vowel for the middle letter?
5. Out of 20 men, in how many different ways can four be chosen to be on guard? In how many of these would one particular man be taken, and from how many would he be left out?
6. Of 12 books of the same size, a shelf will hold 5. How many different arrangements on the shelf may be made?
7. Of 8 men forming a boat's crew, one is selected as stroke. How many arrangements of the rest are possible? When the 4 who row on each side are decided on, how many arrangements are still possible?
8. How many signals may be made with 6 flags of different colors, which can be hoisted either singly, or any number at a time?
9. How many signals may be made with 8 flags of different colors, which can be hoisted either singly, or any number at a time one above another?
10. How many different signals can be made with 10 flags, of which 3 are white, 2 red, and the rest blue, always hoisted all together and one above another?
11. How many signals can be made with 7 flags, of which 2 are red, 1 white, 3 blue, and 1 yellow, always displayed all together and one above another?
12. In how many different ways may the 8 men serving a field-gun be arranged, so that the same man may always lay the gun?

13. Find the number of signals which can be made with 4 lights of different colors when displayed any number at a time, arranged above one another, side by side, or diagonally.
14. From 10 soldiers and 8 sailors, how many different parties of 3 soldiers and 3 sailors can be formed?
15. How many signals can be made with 3 blue and 2 white flags, which can be displayed either singly, or any number at a time one above another?
16. In how many ways can a party of 6 take their places at a round table?
17. Out of 12 Democrats and 16 Republicans, how many different committees can be formed, each consisting of 3 Democrats and 4 Republicans?
18. From 12 soldiers and 8 sailors, how many different parties of 3 soldiers and 2 sailors can be formed?
19. Find the number of combinations of 100 things, 97 together.
20. With 20 consonants and 5 vowels, how many different words can be formed consisting of 3 different consonants and 2 different vowels, any arrangement of letters being considered a word?
21. Of 30 things, how many must be taken together in order that having that number for selection, there may be the greatest possible variety of choice?
22. There are m things of one kind and n of another; how many different sets can be made containing r of the first and s of the second?
23. In how many ways may 10 persons be seated at a round table, so that in no two of the arrangements may every one have the same neighbors?

24. The number of combinations of n things, taken r together, is 3 times the number taken $r-1$ together, and half the number taken $r+1$ together. Find n and r .
25. In how many ways may 12 things be divided into 3 sets of 4?
26. How many words of 6 letters may be formed of 3 vowels and 3 consonants, the vowels always having the even places?
27. From a company of 90 men, 20 are detached for mounting guard each day. How long will it be before the same 20 men are on guard together, supposing the men to be changed as much as possible; and how many times will each man have been on guard?
28. Supposing that a man can place himself in 3 distinct attitudes, how many signals can be made by 4 men placed side by side?
29. How many different arrangements may be made of 11 cricketers, supposing the same two always to bowl?
30. Five flags of different colors can be hoisted either singly, or any number at a time one above another. How many different signals can be made with them?
31. How many signals can be made with 5 lights of different colors, which can be displayed either singly, or any number at a time side by side, or one above another?
32. The number of permutations of n things, 3 at a time, is 6 times the number of combinations, 4 at a time. Find n .
33. At a game of cards, 3 being dealt to each person, any one can have 425 times as many hands as there are cards in the pack. How many cards are there?

BINOMIAL THEOREM.

424. By performing the indicated multiplication

$$(x+a)(x+b)=x^2+(a+b)x+ab.$$

Multiply each side by $x+c$, and the result is

$$\begin{aligned} (x+a)(x+b)(x+c) &= x^3 + (a+b)x^2 + abx \\ &\quad + \quad cx^2 \quad + \quad (ac+bc)x + abc \\ &= x^3 + (a+b+c)x^2 + (ab+ac+bc)x + abc \end{aligned}$$

From this result certain *laws* are to be observed :

I. The number of terms is one more than the number of factors on the left side.

II. The exponent of x in the first term is the same as the number of factors, and decreases by 1 in each succeeding term.

III. The coefficient of x in the first term is unity ;

in the second term, the sum of a, b, c ;

in the third term, the sum of the products two and two, ab, ac, bc ;
and the fourth term is the product abc .

Do the same laws hold, whatever be the number of factors ?

Suppose that these laws hold for r factors, so that

$$(x+a)(x+b)\dots(x+m)=x^r+p_1x^{r-1}+p_2x^{r-2}+p_3x^{r-3}+\dots+p_r$$

where p_1 stands for $a+b+\dots+m$, the sum of the second terms ;

p_2 stands for $ab+ac+\dots$, the sum of the products, two and two ;

p_3 stands for $abc+abd+\dots$, the sum of the products, three and three ;

p_r stands for $abcd\dots$, the product of the r letters.

Multiply by another factor $(x+n)$, and the product of the $r+1$ factors is

$$\begin{aligned} &x^{r+1} + p_1x^r + p_2x^{r-1} + p_3x^{r-2} + \dots + p_rx \\ &\quad + \quad nx^r + p_1nx^{r-1} + p_2nx^{r-2} + \dots + p_{r-1}nx + p_rn \\ &= x^{r+1} + (p_1+n)x^r + (p_2+p_1n)x^{r-1} + (p_3+p_2n)x^{r-2} + \dots + (p_r+p_{r-1}n)x + p_rn \end{aligned}$$

Here the laws I. and II. evidently hold ; and as to the coefficients $p_1+n=a+b+\dots+m+n$, the sum of $r+1$ letters ;

$p_2 + p_1n = (ab + ac + \dots) + (an + bn + \dots + mn)$, the sum of the products, two and two;

$p_3 + p_2n = (abc + abd + \dots) + (abn + acn + \dots)$, the sum of the products, three and three.

$p_r n = abc \dots mn$, the product of the $r + 1$ letters. Hence,

The laws hold for $r + 1$ factors if they hold for r factors.

But they have been shown to hold for *three* factors, therefore they hold for *four* factors, and therefore for *five* factors; and so on for *any* number of factors.*

425. When the factors in the preceding proof are all equal, so that b, c, d, \dots, n , are each equal to a , the left side of the equation becomes

$(x + a)(x + a) \dots$ taken n times; that is, $(x + a)^n$.

On the right side.

$p_1 = a + a + \dots = a$ taken n times $= na$;

$p_2 = aa + aa + \dots = a^2$ taken as many times as there is a choice of 2 letters from n letters,

that is, $p_2 = \frac{n(n-1)}{1 \times 2} a^2$; § 415.

$p_3 = aaa + aaa + \dots = a^3$ taken as many times as there is a choice of 3 letters from n letters,

that is, $p_3 = \frac{n(n-1)(n-2)}{1 \times 2 \times 3} a^3$. § 415.

$p_r n = a \times a \times a \dots = a$ taken n times as a factor $= a^n$.

$\therefore (x + a)^n = x^n + nax^{n-1} + \frac{n(n-1)}{1 \times 2} a^2 x^{n-2} + \frac{n(n-1)(n-2)}{1 \times 2 \times 3} a^3 x^{n-3} + \dots + a^n$.

426. The expression on the right side is called the *expansion* of $(x + a)^n$.

If a and x are interchanged, the expansion will proceed by ascending powers of x , as follows:

$(a + x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{1 \times 2} a^{n-2}x^2 + \dots + nax^{n-1} + x^n$.

* A proof of this kind is called *mathematical induction*.

If $a = 1$, then,

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1 \times 2} x^2 + \dots + nx^{n-1} + x^n.$$

If x is negative, the *odd* powers of x will be *negative* and the *even* powers *positive*.

$$(a-x)^n = a^n - na^{n-1}x + \frac{n(n-1)}{1 \times 2} a^{n-2}x^2 - \frac{n(n-1)(n-2)}{1 \times 2 \times 3} a^{n-3}x^3 + \dots$$

427. It will be observed that the *last factor in the denominator* of the coefficient is one less than the *number of the term*, and is the same as the *exponent of the second letter*; also, that the *last factor of the numerator* of the coefficient is found by subtracting the last factor in the denominator from $n+1$, and that the *exponent of the first letter* is found by subtracting the exponent of the second letter from n . So that,

The *rth* (or general) term in the expansion of $(a+x)^n$ is

$$\frac{n(n-1)\dots(n-r+2)}{1 \times 2 \dots (r-1)} a^{n-r+1} x^{r-1}.$$

Thus, the *third* term of $(a+x)^{20}$ is

$$\frac{20 \times 19}{1 \times 2} a^{18} x^2 = 190 a^{18} x^2.$$

428. The coefficient of the *rth* term from the beginning is equal to the coefficient of the *rth* term from the end.

For, the coefficient of the *rth* term from the beginning is

$$\frac{n(n-1)\dots(n-r+2)}{1 \times 2 \dots (r-1)}, \text{ or } \frac{n(n-1)\dots(n-r+2)}{\underline{r-1}};$$

and this becomes, when both terms are multiplied by $\underline{n-r+1}$,

$$\frac{\underline{n}}{\underline{r-1} \underline{n-r+1}}.$$

The coefficient of the r th term from the end, which is the $(n-r+2)$ th term from the beginning, is

$$\frac{n(n-1)\dots r}{n-r+1};$$

and this also becomes, when both terms are multiplied by $r-1$,

$$\frac{n}{r-1} \frac{r}{n-r+1}.$$

429. It will be evident from § 417, that in the expansion of $(a+x)^n$, the middle term will have the greatest coefficient when n is even; and when n is odd the two middle terms will have *equal* coefficients, and these will be the greatest.

EXERCISE 120.

Expand :

$$1. (1+2x)^5. \quad 3. (2x-3y)^4. \quad 5. \left(1-\frac{3y}{4}\right)^5.$$

$$2. (x-3)^8. \quad 4. (2-x)^3. \quad 6. \left(1-\frac{x}{3}\right)^9.$$

$$7. \text{ Find the fourth term of } (2x-5y)^{12}.$$

$$8. \text{ Find the seventh term of } \left(\frac{x}{2}+\frac{y}{3}\right)^{10}.$$

$$9. \text{ Find the twelfth term of } (a^2-ax)^{15}.$$

$$10. \text{ Find the eighth term of } (5x^2y-2xy^2)^9.$$

$$11. \text{ Find the middle term of } \left(\frac{x}{y}+\frac{y}{x}\right)^8.$$

$$12. \text{ Find the middle term of } \left(\frac{x}{y}-\frac{y}{x}\right)^{10}.$$

$$13. \text{ Find the two middle terms of } \left(\frac{x}{y}-\frac{y}{x}\right)^7.$$

14. Find the r th term of $(2a + x)^n$.
15. Find the r th term from the end of $(2a + x)^n$.
16. Find the $(r + 4)$ th term of $(a + x)^n$.
17. Find the middle term of $(a + x)^{2n}$.
18. Expand $(2a + x)^{12}$, and find the sum of the terms if $a = 1, x = -2$.

WHEN THE EXPONENT IS FRACTIONAL OR NEGATIVE.

430. When n is a whole number, §§ 424–426.

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{1 \times 2} x^2 + \frac{n(n-1)(n-2)}{1 \times 2 \times 3} x^3 + \dots$$

Suppose n to be a positive fraction $\frac{p}{q}$. We may take

$$(1 + x)^{\frac{p}{q}} = 1 + Ax + \dots$$

By raising both sides to the q th power, we have

$$(1 + x)^p = (1 + Ax + \dots)^q,$$

or $(1 + px + \dots) = 1 + qAx + \dots$

Hence, $qA = p$, and therefore $A = \frac{p}{q}$.

That is, *the numerical coefficient of x is the same as the exponent of the binomial, when the exponent is a positive fraction.*

Suppose n is negative $= -m$.

$$\begin{aligned} \text{Then } (1 + x)^n &= (1 + x)^{-m} = \frac{1}{(1 + x)^m} \\ &= \frac{1}{1 + mx + \dots} \quad \text{§§ 424–426.} \\ &= 1 - mx + \dots, \text{ by division.} \end{aligned}$$

That is, *the numerical coefficient of x is the same as the exponent of the binomial when the exponent is negative.*

Therefore, whether n is positive or negative, integral or fractional, in the expansion of $(1+x)^n$ the numerical coefficient of x will be n .

To obtain the coefficients of x^2 , x^3 , etc., in the expansion of $(1+x)^n$, we separate x into two parts, y and z , and write the expression $1+x$ in two ways: first, retaining the parts of x in connection; and secondly, separating them.

We then compare the two results.

First form:

$$\begin{aligned}(1+x)^n &= (1+y+z)^n \\ &= 1 + n(y+z) + B(y+z)^2 + C(y+z)^3 + \dots \\ &= 1 + ny + By^2 + Cy^3 + \dots \\ &\quad + nz + 2Bzy + 3Czy^2 + \dots \\ &\quad + \dots + \dots\end{aligned}$$

Second form:

$$\begin{aligned}(1+x)^n &= (1+y+z)^n = [(1+z)+y]^n \\ &= (1+z)^n \left(1 + \frac{y}{1+z}\right)^n.\end{aligned}$$

In this last equality it is *assumed* that the power of a product is the product of the powers of the factors, whatever be the exponent.

$$\begin{aligned}&(1+z)^n \left(1 + \frac{y}{1+z}\right)^n \\ &= (1+z)^n \left[\left(1 + \frac{ny}{1+z}\right) + \frac{By^2}{(1+z)^2} + \frac{Cy^3}{(1+z)^3} + \dots \right] \\ &= (1+z)^n + n(1+z)^{n-1}y + B(1+z)^{n-2}y^2 + \dots \\ &= 1 + ny + By^2 + \dots \\ &\quad + nz + n(n-1)zy + B(n-2)zy^2 + \dots \\ &\quad + \dots + \dots\end{aligned}$$

Since these two expansions are the expansions of the same thing, in the same form, the coefficients of z , zy , zy^2 , etc., must be the same in both.

In the first expansion :

The coefficient of z is n .

The coefficient of zy is $2B$.

The coefficient of zy^2 is $3C$.

In the second expansion :

The coefficient of z is n .

The coefficient of zy is $n(n-1)$.

The coefficient of zy^2 is $B(n-2)$

Therefore, $n = n$; $2B = n(n-1)$; $3C = B(n-2)$.

Hence, $B = \frac{n(n-1)}{1 \times 2}$; $C = \frac{B(n-2)}{3} = \frac{n(n-1)(n-2)}{1 \times 2 \times 3}$; etc

Accordingly, whether n is positive or negative, integral or fractional,

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1 \times 2} x^2 + \frac{n(n-1)(n-2)}{1 \times 2 \times 3} x^3 + \dots$$

431. In the expansion of $(1+x)^n$, if n is a *positive integer*, the numerator of the last factor of the coefficient of the $(r+1)$ th term, $n-r+1$, will be equal to 0 when $r=n+1$; this term, therefore, and all following terms (for they will also have this factor) will vanish. Hence, the series will *end* with r th, or $(n+1)$ th, term. But if n is *fractional* or *negative*, no value of r will make $n-r+1=0$, and the series will be *infinite*. Hence, the sign = in these cases will mean "*is equal to the limit of the series.*"

432. The sum of the numerical coefficients of the expansion of $(1+x)^n$ is equal to 2^n ; for the sum is equal to $(1+1)^n$, as appears by writing 1 for x in $(1+x)^n$.

433. The sum of the positive coefficients is equal to the sum of the negative coefficients in the expansion of $(1-x)^n$; as appears by writing 1 for x in $(1-x)^n$.

NOTE. The proof in § 430 is taken from the article on Algebra in Encyclopædia Britannica.

Expand to four terms :

(1) $(1+x)^{\frac{1}{2}}$.

$$\begin{aligned}(1+x)^{\frac{1}{2}} &= 1 + \frac{1}{2}x + \frac{\frac{1}{2}(\frac{1}{2}-1)}{1 \times 2}x^2 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{1 \times 2 \times 3}x^3 + \dots \\ &= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{5}{16}x^3 - \dots\end{aligned}$$

(2) $\frac{1}{\sqrt[4]{a^2-2ax}}$.

$$\begin{aligned}\frac{1}{\sqrt[4]{a^2-2ax}} &= (a^2-2ax)^{-\frac{1}{4}} = a^{-\frac{1}{2}} \left\{ 1 - \frac{2x}{a} \right\}^{-\frac{1}{4}}, \\ &= \frac{1}{a^{\frac{1}{2}}} \left\{ 1 + \frac{x}{2a} + \frac{-\frac{1}{4}(-\frac{1}{4}-1)}{1 \times 2} \left(\frac{2x}{a} \right)^2 - \frac{-\frac{1}{4}(-\frac{1}{4}-1)(-\frac{1}{4}-2)}{1 \times 2 \times 3} \left(\frac{2x}{a} \right)^3 \right\} \\ &= \frac{1}{a^{\frac{1}{2}}} \left\{ 1 + \frac{x}{2a} + \frac{5x^2}{8a^2} + \frac{15x^3}{16a^3} + \dots \right\}.\end{aligned}$$

A root may often be extracted by means of an expansion.

(3) Extract the cube root of 344 to six decimal places.

$$\begin{aligned}344 &= 343 \left(1 + \frac{1}{343} \right) = 7^3 \left(1 + \frac{1}{343} \right), \\ \therefore \sqrt[3]{344} &= 7 \left(1 + \frac{1}{343} \right)^{\frac{1}{3}}, \\ &= 7 \left(1 + \frac{1}{3} \times \frac{1}{343} + \frac{\frac{1}{3}(\frac{1}{3}-1)}{1 \times 2} \left(\frac{1}{343} \right)^2 + \dots \right), \\ &= 7 \left(1 + 0.000971815 - 0.000000944 \right), \\ &= 7.006796.\end{aligned}$$

(4) Extract the fifth root of 3128 to six decimal places.

$$\begin{aligned}3128 &= 5^5 + 3 = 5^5 \left(1 + \frac{3}{5^5} \right), \\ \therefore \sqrt[5]{3128} &= 5 \left(1 + \frac{3}{5^5} \right)^{\frac{1}{5}}, \\ &= 5 \left\{ 1 + \frac{1}{5} \times \frac{3}{5^5} + \frac{\frac{1}{5}(\frac{1}{5}-1)}{1 \times 2} \times \left(\frac{3}{5^5} \right)^2 + \dots \right\}, \\ &= 5 \left(1 + 0.000192 - 0.000000073728 + \dots \right), \\ &= 5.000959.\end{aligned}$$

EXERCISE 121.

Expand to four terms :

1. $(1+x)^{\frac{1}{2}}$.

7. $(2x-3y)^{-\frac{1}{2}}$.

2. $(1+x)^{\frac{1}{3}}$.

8. $\sqrt[5]{1-5x}$.

3. $(a+x)^{\frac{1}{2}}$.

9. $\frac{1}{\sqrt{(4a^2-3ax)^3}}$.

4. $(1-x)^{-4}$.

10. $\sqrt[6]{\frac{1}{(1-3y)^{\frac{1}{2}}}}$.

5. $(a^2-x^2)^{\frac{1}{2}}$.

11. $(1+x+x^2)^{\frac{1}{2}}$.

6. $(x^2+xy)^{-\frac{1}{2}}$.

12. $(1-x+x^2)^{\frac{1}{2}}$.

13. Find the r th term of $(a+x)^{\frac{1}{2}}$.14. Find the r th term of $(a-x)^{-3}$.15. Find $\sqrt{65}$ to five decimal places.16. Find $\sqrt[3]{1\frac{1}{30}}$ to five decimal places.17. Find $\sqrt[7]{129}$ to six decimal places.18. Expand $(1-2x+3x^2)^{-\frac{1}{2}}$ to four terms.19. Find the coefficient of x^4 in the expansion of $\frac{(1+2x)^2}{(1+3x)^3}$.20. By means of the expansion of $(1+x)^{\frac{1}{2}}$ show that the limit of the series

$$1 + \frac{1}{2} - \frac{1}{2 \times 2^2} + \frac{1 \times 3}{2 \times 3 \times 2^3} - \frac{1 \times 3 \times 5}{2 \times 3 \times 4 \times 2^4} + \dots \text{ is } \sqrt{2}.$$

CHAPTER XXIII.

CHANCE.

434. If an event can happen in a ways and fail in b ways, and if all these ways are *equally likely* to happen; if, also, *only one can happen*, and *one must happen*, then the mathematical **probability** or **chance** of the event happening is expressed by the fraction

$$\frac{a}{a+b}$$

I. *The probability of an event happening is expressed by the fraction whose numerator is the number of favorable ways, and denominator the whole number of ways.*

Thus, if 1 ball is drawn from a bag containing 3 white balls and 9 black balls, the chance of drawing a white ball is $\frac{1}{12}$; or, as it is expressed, one chance in four.

II. *The probability of an event not happening is expressed by the fraction whose numerator is the number of unfavorable ways, and denominator the whole number of ways.*

Thus if a denote the number of favorable ways, and b the number of unfavorable ways, then the fraction $\frac{b}{a+b}$ will express the probability of the event *not* happening. If, for example, 1 ball is drawn from a bag containing 3 white and 9 black balls, the chance that it will *not* be a white ball is $\frac{9}{12}$.

435. Since
$$\frac{a}{a+b} + \frac{b}{a+b} = 1,$$

it is evident that the chance of an event happening, added to the chance of its not happening, is equal to 1; and,

since an event is *certain* to happen or not happen, it follows that in the theory of chances

III. *Certainty is expressed by unity.*

436. Since $\frac{b}{a+b} = 1 - \frac{a}{a+b}$, it follows that,

IV. *The chance of an event not happening is found by subtracting from unity the chance that it does happen.*

437. If the number of favorable ways is equal to the number of unfavorable ways, then

$$\text{The chance of the event} = \frac{a}{a+b} = \frac{a}{2a} = \frac{1}{2}.$$

This is expressed by saying "the event is as likely to happen as not," or "there is an even chance for the event," or "the odds are even" for and against the event.

Thus, in tossing a cent, there is an even chance that it will fall with the head up.

438. If $a > b$, the chance of the event happening is $> \frac{1}{2}$.

This is expressed by saying "the event is probable," or "the odds are as a to b in favor of the event."

If $a < b$, the chance of the event happening is $< \frac{1}{2}$.

This is expressed by saying "the event is improbable," or "the odds are as b to a against the event."

Thus, the odds are as 5 to 3 in favor of drawing a white ball at the *first trial* from a bag containing 5 white and 3 black balls.

Again, since a die has 6 faces, on one of which is an ace, the chance for an ace the first throw is $\frac{1}{6}$; and the odds are 5 to 1 against an ace.

439. It may be shown that,

V. *If there are several events of which only one can happen, the chance that some of them will happen is the sum of their respective chances of happening.*

For, let a, b, c, \dots denote the number of ways favorable to the first, second, third,..... event respectively; and let p denote the whole number of ways, all equally probable, and of which *one*, and *only one*, must happen. Then the chances of the first, second, third,..... event are

$$\frac{a}{p}, \frac{b}{p}, \frac{c}{p}, \dots \text{ respectively.}$$

Since there are $a + b + c + \dots$ ways favorable to some one or other of the events happening, the chance in favor of some one or other of the events is

$$\frac{a + b + c + \dots}{p} \text{ or } \frac{a}{p} + \frac{b}{p} + \frac{c}{p} + \dots$$

If, for example, a bag contains 3 white, 4 black, 5 red, and 6 green balls, the chance of drawing at the first trial a white or a black ball is $\frac{3}{18} + \frac{4}{18} = \frac{7}{18}$; the chance of drawing a white or a black or a red ball is $\frac{3}{18} + \frac{4}{18} + \frac{5}{18} = \frac{12}{18}$; the chance of drawing a white or a black or a red or a green ball is $\frac{3}{18} + \frac{4}{18} + \frac{5}{18} + \frac{6}{18} = \frac{18}{18} = 1$; that is, *certainty*.

- (1) When two dice are thrown, what is the chance of throwing double aces?

Each die may fall in any one of 6 ways; therefore both dice in $6 \times 6 = 36$ ways (§ 403). Of these ways only one will give double aces. Hence, the chance of double aces = $\frac{1}{36}$. *Ans.*

- (2) What is the chance of throwing doublets in a single throw with two dice?

The dice may fall in 36 ways. Of these, 6 will be doublets. Hence, the chance of throwing doublets = $\frac{6}{36} = \frac{1}{6}$. *Ans.*

- (3) What is the chance of throwing a *six* and a *five* by a single throw of two dice?

The dice may fall in 36 ways. Of these ways the first die may turn up a six and the second a five, or the first may turn up a five and the second a six. Hence, the chance is $\frac{2}{36} = \frac{1}{18}$. *Ans.*

- (4) With two dice, what is the chance of making a throw so that *one* and *only one* die may turn up a *five*?

In 6 of the 36 possible ways one die will turn up a *five*, and the other also will turn up a *five* in 6 ways. Two of these 12 ways will be *double fives*; so that there are 10 ways in which one die, and *only one*, will turn up a *five*, and the chance is $\frac{10}{36}$. *Ans.*

- (5) What is the chance of making a throw that will amount to five?

Of the 36 possible ways, 1 and 4, 4 and 1, 2 and 3, 3 and 2 amount to five. Hence, the chance is $\frac{4}{36} = \frac{1}{9}$. *Ans.*

- (6) In a single throw with two dice, if the player may count the number on one of the dice, or the sum of the numbers on the two dice, what is the chance of throwing five?

The chance is $\frac{1}{6} + \frac{1}{6} + \frac{1}{36}$ (double fives), that is, $\frac{13}{36}$. *Ans.*

- (7) If A's chance of winning a prize is $\frac{1}{8}$, and B's $\frac{1}{8}$, what is the chance that neither will obtain a prize?

The chance that one will win is $\frac{1}{8} + \frac{1}{8} = \frac{1}{4}$. Hence, the chance that neither will win is $1 - \frac{1}{4} = \frac{3}{4}$. *Ans.*

- (8) If 4 cards are drawn from a pack of 52 cards, what is the chance that there will be one of each suit?

Four cards can be selected (\S 415) from the pack in

$$\frac{52 \times 51 \times 50 \times 49}{1 \times 2 \times 3 \times 4} = 270,725 \text{ ways.}$$

But 4 cards can be selected so as to be one of each suit in $13^4 = 28,561$ ways. \S 404.

Hence, the chance is $\frac{28,561}{270,725} = \frac{1}{10}$, nearly.

- (9) If 4 cards are drawn from a pack, what is the chance that they will be the 4 aces?

There are 270,725 ways of drawing four cards, and only one way of drawing the four aces. Hence the chance is $\frac{1}{270,725}$, or 1 chance in 270,725.

- (10) Three balls are to be drawn from an urn containing 5 black, 3 red, and 2 white balls. What is the chance of drawing one red and 2 black balls?

Three balls can be selected from the whole 10 in $\frac{10 \times 9 \times 8}{1 \times 2 \times 3} = 120$ ways. Also, 2 black balls can be selected from the 5 black balls in $\frac{5 \times 4}{1 \times 2} = 10$ ways, and 1 red ball from the 3 red balls in 3

ways. Hence, 1 red and 2 black balls can be drawn in $3 \times 10 = 30$ ways. That is, there are 120 different ways of drawing 3 balls, and 30 of these ways give 1 red and 2 black balls.

The chance, then, of 1 red and 2 black balls is $\frac{30}{120} = \frac{1}{4}$. *Ans.*

- (11) If 2 tickets are drawn from a package of 30 tickets, marked 1, 2, 3,, what is the chance that both will be marked with *odd* numbers ?

Two tickets can be drawn from 30 tickets in $\frac{30 \times 29}{1 \times 2}$ ways; and 2 odd numbers can be drawn from the 15 odd numbers in $\frac{15 \times 14}{1 \times 2}$ ways. Hence, the chance is $\frac{15 \times 14}{30 \times 29} = \frac{7}{29}$. *Ans.*

- (12) In a bag are 5 white and 4 black balls. If they are drawn out one by one, what is the chance that the first will be white, the second black, and so on, alternately ?

The 9 balls can be arranged in $\frac{9!}{2!}$ ways. The 5 white balls can be arranged in the odd places, and the 4 black balls in the even places, in $\frac{5! \times 4!}{1!}$ ways. Hence the chance of alternate order is

$$\frac{\frac{5! \times 4!}{1!}}{\frac{9!}{2!}} = \frac{1}{128}. \text{ Ans.}$$

- (13) From a bag containing 10 balls, 4 are drawn and replaced, then 6 are drawn. Find the chance that the 4 first drawn are among the 6 last drawn.

The second drawing could be made in

$$\frac{10!}{6!4!} = 210 \text{ ways.}$$

But the drawing can be made so as to include the 4 first drawn in

$$\frac{6!}{2!4!} = 15 \text{ ways.}$$

since the only choice consists in selecting 2 balls from the 6 not previously drawn. Hence, the chance is $\frac{15}{210} = \frac{1}{14}$. *Ans.*

- (14) The chance of an event is $\frac{4}{7}$. What are the odds in favor of the event?

4 to 3. *Ans.*

- (15) The odds against an event are 3 to 1. What is the chance of the event?

$\frac{1}{4}$. *Ans.*

- (16) The odds against an event are m to n . What is the chance of the event?

$$\frac{n}{m+n}. \text{ Ans.}$$

- (17) If 4 coppers are tossed, what is the chance that exactly 2 will turn up heads?

Since each coin may fall in two ways, the 4 coins may fall in $2^4 = 16$ ways. The 2 coins to turn up heads can be selected from the 4 coins in $\frac{4 \times 3}{1 \times 2} = 6$ ways. Hence, the chance is $\frac{6}{16} = \frac{3}{8}$; and the odds are 5 to 3 against it.

- (18) A has 3 tickets in a lottery where there are 3 prizes and 6 blanks. Find his chance of winning *one* prize, *two* prizes, *three* prizes, respectively.

Three tickets can be selected from 9 tickets in $\frac{9 \times 8 \times 7}{1 \times 2 \times 3} = 84$ ways. A prize ticket can be selected from the 3 prize tickets in 3 ways, and 2 blanks can be selected from the 6 blanks in $\frac{6 \times 5}{1 \times 2} = 15$ ways; therefore, 1 prize and 2 blank tickets can be selected in $3 \times 15 = 45$ ways. Hence, the chance of drawing *one* prize is $\frac{3}{8}$.

Again, 1 blank and 2 prize tickets can be selected in $6 \times \frac{3 \times 2}{1 \times 2} = 18$ ways. Hence the chance of *two* prizes is $\frac{3}{8}$.

Also, the 3 prize tickets can be selected in only 1 way. Hence, the chance of drawing *three* prizes is $\frac{1}{84}$.

- (19) What is the chance that A in Ex. (18) wins *at least* one prize?

The chance is $\frac{4}{4} + \frac{1}{4} + \frac{1}{4} = \frac{6}{4} = \frac{3}{2}$. For, he will have at least one prize in any one of the three cases given in (18).

Or, the chance may be found in this way: A gets a prize unless his three tickets all turn out blanks. Three tickets can be selected from the whole number in 84 ways, and from the 6 blanks in $\frac{6 \times 5 \times 4}{1 \times 2 \times 3} = 20$ ways. Hence, the chance that they will all be blanks is $\frac{20}{84} = \frac{5}{21}$; and the chance against this result is $1 - \frac{5}{21} = \frac{16}{21}$.

440. If a person is to receive a prize in case a particular event happens, the sum of money for which he may equitably sell his chance for the prize is called his **expectation** from the event.

VI. *The expectation from an uncertain event is the product of the chance that the event will happen by the sum to be realized in case the event happens.*

Thus, if there is a lottery with 40 tickets, and 1 prize worth \$100, a person might equitably pay \$100 for the whole 40 tickets, since one of them is sure to draw the \$100. Now, all of the tickets are of equal value before the drawing; hence the value of each ticket is $\frac{1}{40}$ of \$100. The value of 5 tickets is $\frac{5}{40}$ of \$100, or \$12.50; that is, the product of the chance which the holder of 5 tickets has of winning the prize and the value of the prize; that is, $\frac{5}{40}$ of \$100.

- (20) If a lottery has 1 prize of \$50, 2 prizes of \$5 each, 4 prizes of \$1 each, and 13 blanks, what is the expectation of the holder of 1 ticket?

The chance of drawing the prize of \$50 is $\frac{1}{40}$, and the expectation is $\frac{1}{40}$ of \$50 = \$2.50. The chance of drawing a prize of \$5 is $\frac{2}{40}$, and the expectation is $\frac{2}{40}$ of \$5 = \$0.50. The chance of drawing a prize of \$1 is $\frac{4}{40}$, and the expectation is $\frac{4}{40}$ of \$1 = \$0.20. Hence, the whole expectation is \$2.50 + \$0.50 + \$0.20 = \$3.20. *Ans.*

EXERCISE 122.

1. If I throw a single die, what is the chance that it will turn up:
 - (i.) An ace?
 - (ii.) An ace or a two?
 - (iii.) Neither an ace nor a two?
2. The chance of a plan succeeding is $\frac{1}{4}$. What is the chance that it fails?
3. If the odds are 10 to 1 against an event, what is the probability of its happening?
4. If the odds are 5 to 2 in favor of the success of an experiment, what are the respective chances of success or failure?
5. The chance of an event is $\frac{3}{8}$. Find the odds for or against the event.
6. What is the chance of a year, not a leap-year, having 53 Sundays?
7. Two numbers are chosen at random. Find the chance that their sum is even.
8. If 4 cards are drawn from a pack, what is the chance that they will all be hearts?
9. If 10 persons stand in a line, what is the chance that 2 assigned persons will stand together?
10. If 10 persons form a ring, what is the chance that 2 assigned persons will stand together?
11. Show that, if n persons sit down at a round table, the odds against 2 particular persons sitting next to each other are $n-3$ to 2.

12. If 2 letters are selected at random out of the alphabet, what is the chance that both will be vowels ?
13. Five men, A, B, C, D, E, speak at a meeting, and it is known that A speaks before B. What is the chance that A speaks *immediately* before B ?
14. A, B, C have equal claims for a prize. A says to B, "You and I will draw lots, and the winner shall draw lots with C for the prize." Is this fair ?
15. A person is allowed to draw 2 tickets from a bag containing 40 blank tickets, and 10 tickets each entitling the holder to a prize of \$100. What is his expectation ?
16. One of two events must happen. If the chance of one is $\frac{2}{3}$ of that of the other, find the odds on the first.
17. There are 3 events, A, B, C, one of which must happen. The odds are 3 to 8 on A, and 2 to 5 on B. Find the odds on C.
18. In a bag are 7 white and 5 red balls. Find the chance that if one is drawn it will be (i.) white or (ii.) red ; or, if two are drawn, that they will be (i.) both white, (ii.) both red, or (iii.) one white and the other red.
19. If 3 cards are drawn from a pack, what is the chance that they will be king, queen, and knave of the same suit ?
20. A general orders 2 men by lot out of 100 mutineers to be shot ; the real leaders of the mutiny being 10 in number. Find the chance (i.) that one only, (ii.) that two, of the leaders will be shot.
21. Show that the odds are 8 to 1 against throwing 9 in a single throw with 2 dice.

22. Show that in a throw with 3 dice the chance of either a triplet or a doublet is $\frac{1}{8}$.
23. In a bag are 5 white and 4 black balls. If drawn out, one by one, what is the chance that the first will be white, the second black, and so on, alternately ?
24. A bag contains 2 white balls, 3 black balls, and 5 red balls. If 4 balls are drawn, find the chance that there shall be among them :
- (i.) Both the white balls.
 - (ii.) Two *only* of the black balls.
 - (iii.) Two *at least* of the red balls.

441. A series of events, such that *only one* of them can happen, may be called a series of **exclusive**, or **dependent**, events.

Two or more events, such that *both* or *all* may happen, are called **non-exclusive**, or **independent**, events.

Thus, if a copper be thrown twice in succession it may fall head up *both* times ; and, if it be thrown ten times, it is possible for it to fall head up each time.

442. If there are two or more independent events, the occurrence of all of them simultaneously or in succession may be regarded as a single **compound** event.

Thus, in tossing a copper twice, the event of its falling with head up at both trials may be regarded as an event *compounded of two simple* events ; namely, with head up at the first trial, and with head up at the second trial.

- (1) In tossing a copper twice, what is the chance of its falling head up both times ?

The chance of a head at each trial is $\frac{1}{2}$. If these separate chances were added (according to Rule V.), the result would be 1 ; that is, certainty ; a result obviously false. Rule V. applies only to *dependent* or *exclusive* events. In this case, however, the events are *independent*, or *non-exclusive*.

Now, each time the copper is thrown, it can fall in two ways. Hence, the *double fall* can occur in $2 \times 2 = 4$ ways: § 403.

1. Both times a head.
2. First time a head, second time a tail.
3. First time a tail, second time a head.
4. Both times a tail.

Only *one* of these *four* ways gives *heads both times*. Hence, the chance of heads *both* times is $\frac{1}{4} = \frac{1}{2} \times \frac{1}{2}$; that is, the product of the separate chances of a head at each trial.

In general,

VII. *The chance that two independent events both happen is the product of their separate chances of happening.*

For the product of the denominators of the separate chances is the whole number of ways in which the compound event can happen; and the product of the numerators is the number of ways favorable to its happening.

- (2) A bag contains 3 balls, two of which are white; another contains 6 balls, five of which are white. If a person draws 1 ball from each bag, what is the chance that *both* balls drawn will be white?

The first ball can be drawn in 3 ways and the second in 6 ways. Hence, both can be drawn in $3 \times 6 = 18$ ways. Also, the first ball can be a *white* ball in 2 ways, and the second in 5 ways. Hence, they can be *both white* in $2 \times 5 = 10$ ways. The chance of *both white* therefore is $\frac{10}{18} = \frac{5}{9} = \frac{2}{3} \times \frac{5}{6}$; that is, the product of the separate chances of a white ball at each trial.

443. In like manner, whatever the number of simple events that unite to produce a compound event, it may be shown that:

VIII. *The chance of a compound event is the product of the separate chances of the simple events that unite to produce it.*

NOTE. It is important not to confound exclusive events with non-exclusive, and not to apply Rule V. to problems to which Rule VII. applies.

- (3) The chance that A can solve a given problem is $\frac{3}{4}$, and the chance that B can solve it is $\frac{5}{12}$. If both try, what are the chances (i.) that both solve it; (ii.) that A solves it and B fails; (iii.) that A fails and B solves it; (iv.) that both fail?

A's chance of success is $\frac{3}{4}$, A's chance of failure is $\frac{1}{4}$.

B's chance of success is $\frac{5}{12}$, B's chance of failure is $\frac{7}{12}$.

Therefore, the chance of (i.) is $\frac{3}{4} \times \frac{5}{12} = \frac{15}{48}$;

the chance of (ii.) is $\frac{3}{4} \times \frac{7}{12} = \frac{21}{48}$;

the chance of (iii.) is $\frac{1}{4} \times \frac{5}{12} = \frac{5}{48}$;

the chance of (iv.) is $\frac{1}{4} \times \frac{7}{12} = \frac{7}{48}$.

The sum of these four chances is $\frac{15}{48} + \frac{21}{48} + \frac{5}{48} + \frac{7}{48} = 1$, as it ought to be, since 1 of the 4 results is *certain* to happen.

- (4) In Ex. (3) what is the chance that the problem will be solved?

The chance that *both fail* is $\frac{7}{48}$. Hence, the chance that *both do not fail*, or that the problem will be solved, is $1 - \frac{7}{48} = \frac{41}{48}$.

- (5) There are 3 bags, the first containing 1 white, and 1 black ball; the second, 1 red and 2 white balls; the third, 3 white and 2 green balls. If a person draw a ball from each bag, what is the chance that all three balls drawn will be white?

$\frac{1}{2} \times \frac{2}{3} \times \frac{3}{5} = \frac{1}{5}$. *Ans.*

- (6) Under the conditions of the last problem, what is the chance that no one of the balls drawn will be white?

The chances of failing to draw a white ball at the three trials are $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{5}$, respectively. Therefore, the chance of failing altogether is $\frac{1}{2} \times \frac{1}{3} \times \frac{2}{5} = \frac{1}{15}$. *Ans.*

- (7) What is the chance in Ex. (5) of drawing *at least one* white ball?

One white ball will be drawn unless all three trials fail. The chance that all three fail is $\frac{1}{15}$. Therefore, the chance of drawing at least one white ball is $1 - \frac{1}{15} = \frac{14}{15}$. *Ans.*

- (8) What is the chance in Ex. (5) that *one*, and *only one*, white ball should be drawn in the three trials?

The chance of a white ball from the first bag and not from the others is $\frac{1}{3} \times \frac{1}{3} \times \frac{2}{3} = \frac{2}{27}$

The chance of a white ball from the second bag and not from the others is $\frac{1}{3} \times \frac{2}{3} \times \frac{1}{3} = \frac{2}{27}$

The chance of a white ball from the third bag and not from the others is $\frac{1}{3} \times \frac{1}{3} \times \frac{2}{3} = \frac{2}{27}$

Therefore, the sum of these chances is $\frac{6}{27} = \frac{2}{9}$

- (9) When 6 coins are tossed, what is the chance that *at least one* will fall with the head up?

The chance that *all* will fall heads down is $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{64}$. Hence the chance that this will not happen is $1 - \frac{1}{64} = \frac{63}{64}$.

- (10) When 6 coins are tossed, what is the chance that *one*, and *only one*, will fall with the head up?

The chance that the first alone falls with heads up is $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{64}$; the chance that the second alone falls with the head up is $\frac{1}{64}$, and so on.

Hence, the chance that some one, and only one, falls head up is $\frac{1}{64} + \frac{1}{64} + \frac{1}{64} + \frac{1}{64} + \frac{1}{64} + \frac{1}{64} = \frac{6}{64} = \frac{3}{32}$. *Ans.*

- (11) When 4 dice are thrown, what is the chance that *two*, and *only two*, turn up aces?

The chance that any particular two of the 4 dice turn up aces, and the other two something else, is $\frac{1}{6} \times \frac{1}{6} \times \frac{5}{6} \times \frac{5}{6} = \frac{25}{324}$.

Now the number of ways in which this can happen is the number of ways in which two dice can be selected from 4 dice, or 6 ways. Hence, the chance that *two*, and *only two*, turn up aces is $6 \times \frac{25}{324} = \frac{25}{54}$. *Ans.*

- (12) When 4 dice are thrown, what is the chance that they will all turn up alike?

The chance that the first and second turn up alike is $\frac{1}{6}$.

The chance that the third turns up like the first and second is $\frac{1}{6}$.

The chance that the fourth turns up like the others is $\frac{1}{6}$.

Hence, the chance that the four turn up alike is $\frac{1}{216}$. *Ans.*

- (13) When 4 dice are thrown, what is the chance that two, and only two of them, should turn up alike?

The chance that any two should be alike is $\frac{1}{6}$, the chance that the third should be different is $\frac{5}{6}$, and the chance that the fourth should be different from all the rest is $\frac{4}{6}$. Hence, the chance that a pair should agree while the others should differ from the pair and from each other is $\frac{1}{6} \times \frac{5}{6} \times \frac{4}{6} = \frac{20}{216}$. The pair to agree may be selected in $\frac{4 \times 3}{1 \times 2} = 6$ ways. Hence, the total chance is $6 \times \frac{20}{216} = \frac{5}{9}$. *Ans.*

- (14) When 4 dice are thrown, what is the chance that they should all fall different?

The chance that the second should differ from the first is $\frac{5}{6}$, that the third should differ from both the first and the second is $\frac{4}{6}$, and that the fourth should differ from all the others is $\frac{3}{6}$. Hence, the required chance is $\frac{5}{6} \times \frac{4}{6} \times \frac{3}{6} = \frac{5}{18}$. *Ans.*

- (15) A single die is thrown until it turns up an ace. What is the chance that it must be thrown *at least* 10 times? What is the chance that it must be thrown *exactly* 10 times?

The chance of failing the first 9 times is (Rules IV. and VII.) $(\frac{5}{6})^9$. This, then, is the chance that *at least* 10 trials must be made. Since $(\frac{5}{6})^9$ is the chance of failing the first 9 trials, and $\frac{1}{6}$ the chance of success the next trial; therefore (Rule VII) $(\frac{5}{6})^9 \times \frac{1}{6}$ is the chance that *exactly* 10 throws must be made.

- (16) What is the chance that a person with 2 dice will throw double aces exactly 3 times in 5 trials?

The chance of throwing double aces at any particular trial is $\frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$, and of failing is $\frac{35}{36}$. Hence, the chance of succeeding at 3 assigned trials, and failing at the other 2 trials, is $(\frac{1}{36})^3 \times (\frac{35}{36})^2$. Now double aces will be thrown *exactly three* times if thrown in any set of 3 trials that may be assigned out of the 5 trials, and fail in the other 2 trials. 3 trials can be assigned out of 5 trials in $\frac{5 \times 4 \times 3}{1 \times 2 \times 3} = 10$ ways. Hence, the chance is $(\frac{1}{36})^3 \times (\frac{35}{36})^2 \times 10$.

- (17) A and B throw with a single die alternately, A throwing first; and the one who throws an ace first is to receive a prize of \$10. What are their respective expectations?

The chance for the prize at the first throw is $\frac{1}{6}$; at the second, $\frac{5}{6} \times \frac{1}{6}$; at the third, $(\frac{5}{6})^2 \times \frac{1}{6}$; at the fourth, $(\frac{5}{6})^3 \times \frac{1}{6}$; and so on.

As A has the first, third, etc., and B the second, fourth, etc., throws,

$$\frac{\text{A's chance}}{\text{B's chance}} = \frac{\frac{1}{6} + (\frac{5}{6})^2 \text{ of } \frac{1}{6} + \dots}{\frac{5}{6} \text{ of } \frac{1}{6} + (\frac{5}{6})^3 \text{ of } \frac{1}{6} + \dots} = \frac{5}{4}; \text{ so that,}$$

A's expectation is $\frac{5}{9}$ of \$10 = \$5 $\frac{5}{9}$, and

B's expectation is $\frac{4}{9}$ of \$10 = \$4 $\frac{4}{9}$.

- (18) A and B play at a game that cannot be a drawn game, and on an average A wins 3 games out of 5 games. Out of 5 games what is the chance that A wins *at least three*?

The chance that A wins 3 assigned games out of 5 games is $(\frac{3}{5})^3 \times (\frac{2}{5})^2 = \frac{108}{3125}$. The 3 games may be assigned in 10 ways. Hence, A's chance for 3 games is $10 \times \frac{108}{3125} = \frac{1080}{3125}$.

The chance that A wins 4 games, and B the other game, is $\frac{162}{3125} \times 5 = \frac{810}{3125}$. The chance that A wins *all* the games is $(\frac{3}{5})^5 = \frac{243}{3125}$.

For A to win *at least* 3 games, he must win 3, 4, or 5 games. Hence, A's chance for at least 3 games is $\frac{1080}{3125} + \frac{810}{3125} + \frac{243}{3125} = \frac{2133}{3125}$.

- (19) A's skill at a game, which cannot be a drawn game, is to B's skill as 3 to 4. If they play 3 games, what is the chance that A will win more games than B?

Their respective chances of winning a particular game are $\frac{3}{7}$ and $\frac{4}{7}$. For A to win more games than B, he must win all 3 games or 2 games. The chance that A wins all three is $(\frac{3}{7})^3 = \frac{27}{343}$. The chance that A wins any assigned set of 2 games out of the 3 games, and that B wins the other, is $(\frac{3}{7})^2 \times \frac{4}{7}$. As there are 3 ways of assigning a set of 2 games out of 3, the chance that A wins 2 games, and B the other, is $(\frac{3}{7})^2 \times \frac{4}{7} \times 3 = \frac{108}{343}$. Hence, the chance that A wins more than B is $\frac{27}{343} + \frac{108}{343} = \frac{135}{343}$.

- (20) In the last example, find B's chance of winning more games than A.

B's chance of winning all three games is $(\frac{1}{3})^3 = \frac{1}{27}$. His chance of winning 2 games, and A the other game, is $(\frac{1}{3})^2 \times \frac{1}{3} \times 3 = \frac{1}{9}$. Hence, his chance of winning more games than A is $\frac{1}{27} + \frac{1}{9} = \frac{4}{27}$.

Notice that A's chance added to B's chance, $\frac{1}{27} + \frac{4}{27} = 1$. Why should this be so?

- (21) A plays a set of games (drawn games excluded) with B, his chance of winning a single game being to B's as 3:2. What is the probability:

- (i.) That A will win 4 games at least out of 7?
 (ii.) That he will win 4 games before B wins 3?

(i.) A's chance of winning a single game is $\frac{3}{5}$, and B's chance is $\frac{2}{5}$. The chance that A wins *at least* 4 games out of 7 is the sum of the chances that he wins 4, 5, 6, or 7 games out of 7, *in any possible order*.

The chance that he wins all 7 games = $(\frac{3}{5})^7$.

The chance that he wins 6 games = $7 (\frac{3}{5})^6 \times \frac{2}{5}$.

The chance that he wins 5 games = $21 (\frac{3}{5})^5 \times (\frac{2}{5})^2$.

The chance that he wins 4 games = $35 (\frac{3}{5})^4 \times (\frac{2}{5})^3$.

The sum of these values = $\frac{11097}{5^5}$. *Ans.*

(ii.) Here the chance required is that A shall win at least 4 games; that is, 4, 5, or 6 games out of 6.

The chance that he wins all 6 games = $(\frac{3}{5})^6$.

The chance that he wins 5 games = $6 \times (\frac{3}{5})^5 \times \frac{2}{5}$.

The chance that he wins 4 games = $15 \times (\frac{3}{5})^4 \times (\frac{2}{5})^2$.

The sum of these values = $\frac{1701}{5^5}$. *Ans.*

- (22) In a certain locality it is found that, on the average for 10 years, out of 100 persons 40 years old at the beginning of the decade, 20 die; out of 100 persons 50 years old, 30 die; and out of 100 persons 60 years old, 40 die. What is the odds against a person 40 years old living 30 years longer?

The chance that he dies between 40 and 50 is $\frac{1}{3}$; that he lives till 50, and dies between 50 and 60, is $\frac{2}{3} \times \frac{1}{10} = \frac{2}{30}$; that he lives till 60, and dies between 60 and 70, is $\frac{2}{3} \times \frac{1}{10} \times \frac{1}{10} = \frac{2}{300}$. Hence, the chance that he dies between 40 and 70 is $\frac{1}{3} + \frac{2}{30} + \frac{2}{300} = \frac{100}{300}$. Therefore, the odds against his living for 30 years are 83 to 42, or about 2 to 1.

- (23) A is 40 years old and B 50 years old. What is the probability that at least one of them will be alive 10 years hence?

The chance that A dies is $\frac{1}{3}$, and the chance that B dies is $\frac{1}{10}$. Hence, the chance that both die is $\frac{1}{3} \times \frac{1}{10} = \frac{1}{30}$; and the chance that one at least will be alive is $1 - \frac{1}{30} = \frac{29}{30}$.

444. Cases often occur where the simple events which unite to form the compound event are so related that the happening of one of them alters the chances of the others.

- (1) What is the chance of drawing in succession 2 vowels from the alphabet?

The chance of drawing a vowel the first time is $\frac{5}{26}$; but, if one vowel is drawn, the chance of drawing another is $\frac{4}{25}$. Hence, (Rule VII.) the required chance is $\frac{5}{26} \times \frac{4}{25} = \frac{2}{13}$.

- (2) A bag contains 5 white and 6 black balls. What is the chance of drawing 5 times in succession a white ball, the balls drawn not being replaced?

$$\frac{5}{11} \times \frac{4}{10} \times \frac{3}{9} \times \frac{2}{8} \times \frac{1}{7} = \frac{1}{112}.$$

- (3) What would have been the chance in the last example if after each drawing the ball had been replaced?

$$\left(\frac{5}{11}\right)^5.$$

- (4) If the chance of an event at first is as a to b , and if whenever it happens, the number of favorable ways, as well as the whole number of ways, is diminished by unity; find the chance that the event will occur n times in succession.

$$\frac{a(a-1)(a-2)\dots(a-n+1)}{b(b-1)(b-2)\dots(b-n+1)}.$$

Ans.

- (5) A bag contains 5 white and 6 black balls. If 5 balls are drawn in succession, and no one of them replaced, what is the probability that the first three will be white, and the fourth and fifth black?

The separate chances for the 5 simple events are respectively $\frac{5}{11}$, $\frac{4}{10}$, $\frac{3}{9}$, $\frac{6}{8}$, $\frac{5}{7}$. Hence, the chance for the compound event is $\frac{5}{11} \times \frac{4}{10} \times \frac{3}{9} \times \frac{6}{8} \times \frac{5}{7} = \frac{5}{154}$. *Ans.*

- (6) Find the probability in the last example that the 5 balls drawn will be 3 white and 2 black balls.

Here the chance required is that 3 white and 2 black should be drawn not in any *assigned order*, as in the last case, but in *any possible order*. Now 5 things, of which 3 are alike and the other 2 alike, may be arranged (§ 411) in

$$\frac{|5|}{|3|2|} = 10 \text{ ways.}$$

Hence, the probability is $10 \times \frac{5}{154} = \frac{5}{77}$.

- (7) Find the respective probabilities in Examples (5) and (6) if after each drawing the ball is replaced.

In Ex. (5), $(\frac{5}{11})^3 \times (\frac{6}{11})^2$. *Ans.*

In Ex. (6), $10 \times (\frac{5}{11})^3 \times (\frac{6}{11})^2$. *Ans.*

- (8) A purse contains 9 silver dollars and 1 gold eagle, and another contains 10 silver dollars. If 9 coins are taken out of the first purse and put into the second, and then 9 coins are taken out of the second and put into the first purse, which purse now is the more likely to contain the gold coin?

The gold eagle will not be in the second purse unless it (i.) was among the 9 coins taken out of the first and put into the second purse, (ii.) and *not* among the 9 coins taken out of the second and put into the first purse. The chance of (i.) is $\frac{9}{10}$, and when (i.) has happened the chance of (ii.) is $\frac{1}{9}$. Hence, the chance of *both* happening is $\frac{9}{10} \times \frac{1}{9} = \frac{1}{10}$. Therefore, the chance that the eagle is in the second purse is $\frac{1}{10}$, and the chance that it is in the first purse is $1 - \frac{1}{10} = \frac{9}{10}$. Since $\frac{9}{10}$ is greater than $\frac{1}{10}$, therefore the gold coin is more likely to be in the first purse.

- (9) In a bag are 2 red and 3 white balls. A is to draw a ball, then B, and so on alternately; and whichever draws a white ball first is to receive \$10. Find their expectations.

A's chance of drawing a *white* ball at the first trial is $\frac{3}{5}$. B's chance of *having a trial* is equal to A's chance of drawing a *red* ball = $\frac{2}{5}$. In case A drew a red ball there would be 1 red and 3 white balls left in the bag, and B's chance of drawing a white ball would be $\frac{3}{4}$. Hence, B's chance of having the trial and drawing a white ball is $\frac{2}{5} \times \frac{3}{4} = \frac{3}{10}$; and B's chance of drawing a red ball is $\frac{2}{5} \times \frac{1}{4} = \frac{1}{10}$.

A's chance of *having a second trial* is equal to B's chance of drawing a *red* ball = $\frac{1}{10}$. In case B drew a red ball there would be 3 white balls left, and A's chance of drawing a white ball would be *certainty*, or 1.

A's chance, therefore, is $\frac{3}{5} + \frac{1}{10} = \frac{7}{10}$; and B's chance is $\frac{3}{10}$.

A's expectation, then, is \$7, and B's \$3.

445. In general, when it is required to find which of two doubtful events is more likely to happen, it is necessary to find their respective chances, and then to compare the results obtained.

- (1) In one throw with two dice which sum is more likely to be thrown, 9 or 12?

Out of the 36 possible ways of falling, *four* give the sum 9 (namely, 6 + 3, 3 + 6, 5 + 4, 4 + 5), and *only one* way gives 12 (namely, 6 + 6). Hence, the chance of throwing 9 is *four times* as good as that of throwing 12.

- (2) With three dice what are the relative chances of throwing a doublet and a triplet?

The chance of throwing a doublet is

$$3 \times \frac{6 \times 1 \times 5}{6^3} = \frac{5}{8}.$$

The chance of throwing a triplet is

$$\frac{6 \times 1 \times 1}{6^3} = \frac{1}{36}.$$

Hence, the chance of a doublet is 15 times that of a triplet.

- (3) A bag contains 1 black and 4 white balls, and another bag contains 7 black and 3 white balls. If a person draws a ball from one of the bags, (i.) what is the chance that it be a white ball? (ii.) what is the ratio of the chance of its being drawn from the first bag to that of its being drawn from the second bag?

The person (so far as we know) is as likely to choose one bag as the other. Hence, the chance of his choosing the first bag is $\frac{1}{2}$; and the chance of his drawing a white ball from the first bag is $\frac{4}{5}$. Therefore, the chance of drawing a white ball from the first bag is $\frac{1}{2} \times \frac{4}{5} = \frac{2}{5}$. In the same way, the chance of drawing a white ball from the second bag is found to be $\frac{1}{2} \times \frac{3}{10} = \frac{3}{20}$.

Therefore, the chance of drawing a white ball is $\frac{2}{5} + \frac{3}{20} = \frac{11}{20}$; and the ratio of the separate chances is 8 : 3.

- (4) Suppose in the last example that at the first trial a white ball is actually drawn. What are now the chances that it came from the first bag, and from the second, respectively?

Let x and y represent the chances required.

$$\text{Then, by Ex. (3),} \quad \frac{x}{y} = \frac{8}{3}.$$

$$\text{Also,} \quad x + y = 1,$$

since the ball must have come from one or the other bag.

The solution of these equations gives

$$x = \frac{8}{11}, \quad \text{and} \quad y = \frac{3}{11}.$$

446. From Examples (3) and (4) it will be seen that,

IX. If a doubtful event may happen in some one of several ways, the actual happening of the event changes its probability, and the separate probabilities of the several ways of happening, in the same ratio; and this ratio is the reciprocal of the fraction that expresses the chance of the event before it actually happens.

Thus, in Ex. (3), the chance of the event *before it happened*, and the chance of the two separate ways of happening, were found to be

$\frac{1}{20}, \frac{2}{20}, \frac{3}{20}$; in Ex. (4) it was shown that *after the event happened* these chances became $1, \frac{1}{11}, \frac{2}{11}$, respectively. The values $1, \frac{1}{11}, \frac{2}{11}$ are obtained by multiplying $\frac{1}{20}, \frac{2}{20}, \frac{3}{20}$ by $\frac{20}{11}$; that is, by *the reciprocal of $\frac{1}{20}$* .

Evidently, the happening of the event must change to unity the chance of the event; and must therefore increase to unity the sum of the separate chances.

So long, however, as the only *additional* knowledge about the event is the fact that it has happened, the *relative* probabilities of the separate ways of happening remain unchanged. Therefore, the several fractions which before expressed the probabilities of the separate ways of happening must now be multiplied by the same factor, and that factor is the reciprocal of the fraction that expressed the probability of the event before it happened.

EXERCISE 123.

1. The chance that A can solve a certain problem is $\frac{1}{4}$, and the chance that B can solve it is $\frac{3}{4}$. What is the chance that the problem will be solved if both try?
2. What is the chance of throwing at least one ace in 2 throws with one die?
3. If n coins are tossed up, what is the chance that one, and only one, will turn up head?
4. What is the chance of throwing double sixes at least once in 3 throws with 2 dice?
5. A copper is tossed 3 times. Find the odds that it will fall :
(i.) One head and two tails without regard to order.
(ii.) Head, tail, head.
6. If a copper is tossed 4 times, find the odds that it will fall 2 heads and 2 tails sooner than 4 heads.
7. If from a lottery of 30 tickets, marked 1, 2, 3,, four tickets are drawn, what is the chance that 1 and 2 will be among them?

8. If 2 coppers are tossed 3 times, find the odds that they will fall 2 heads and 4 tails.
9. There are 10 tickets, five of which are numbered 1, 2, 3, 4, 5, and the other five are blank. Find the chance that the sum of the numbers on the tickets drawn in 3 trials will be 10, one ticket being drawn and then replaced at each trial?
10. Find the chance in Ex. 9 if the tickets are not replaced.
11. A bag contains 4 white and 6 red balls. A, B and C draw each a ball, in order, replacing. Find the chance that they have drawn :
 - (i.) Each a white ball.
 - (ii.) A and B white, C red.
 - (iii.) Two white and one red.
12. Find the answer to Ex. 11 if the balls are not replaced.
13. A draws 4 times from a bag containing 2 white and 8 black balls, replacing. Find the chance that he has drawn :
 - (i.) Two white, two black.
 - (ii.) Not less than two white.
 - (iii.) Not more than two white.
 - (iv.) One white, three black.
14. Find the odds against throwing one of the two numbers 7 or 11 in a single throw with 2 dice.
15. If a copper is tossed 5 times, what is the chance that it will fall heads either 2 times or else 3 times?
16. Find the same chance if the copper is tossed 6 times.
17. In one bag are 10 balls and in another 6; and in each bag the balls are marked 1, 2, 3, etc. What is the chance that on drawing one ball from each bag the two balls will have the same number?

18. A bag contains n balls. A person takes out one ball and then replaces it. He does this n times. What is the chance that he has had in his hand every ball in the bag?
19. If on an average 9 ships out of 10 return safe to port, what is the chance that out of 5 ships expected at least 3 will return?
20. What is the chance of throwing double sixes at least once in 3 throws with a pair of dice?
21. What is the chance of throwing 15 in one throw with 3 dice?
22. In 5 throws with a single die what is the chance of throwing an ace:
 - (i.) Three times exactly?
 - (ii.) Not less than three times?
 - (iii.) Not more than three times?
23. In a bag are 3 white, 5 red, and 7 black balls, and a person draws three times, replacing. Find the chance that he has drawn:
 - (i.) A ball of each color.
 - (ii.) Two white, one red.
 - (iii.) Three red.
 - (iv.) Two red, one black.
24. A and B play at chess, and A wins on an average 2 games out of 3. Find the chance of A's winning exactly 4 games out of the first 6, drawn games being disregarded.
25. A and B engage in a game in which A's skill is to B's as 2:3. Find the chance of A's winning at least 2 games out of the first 5, drawn games not being counted.

26. The skill of A is double that of B. Find the odds against A's winning 4 games before B wins 2.
27. If B's skill in a certain game is equal to three-fifths of A's, find A's chance of winning 5 games out of 8.
28. A bag contains 4 red balls and 2 others, each of which is equally likely to be red or white. Three times in succession a ball is drawn and replaced. Find the chance that all the drawn balls are red.
29. A man has left his umbrella in one of 3 shops which he visited in succession. He is in the habit of leaving it, on an average, once every 4 times that he goes to a shop. Find the chance that he left it in the first, second, and third shops, respectively.
30. A bets B \$10 to \$1 that he will throw heads at least once in 3 trials. What is B's expectation? What would have been a fair bet?
31. A draws 5 times (replacing) from a bag containing 3 white and 7 black balls; every time he draws a white ball he is to receive \$1, and every time he draws a black ball he is to pay 50 cents. What is his expectation?
32. From a bag containing 2 eagles, 3 dollars, and 3 quarter-dollars, A is to draw one coin and then B three coins; and A, B, and C are to divide equally the value of the remainder. What are their expectations?
33. A, B, and C, staking each \$5, draw from a bag in which are 4 white and 6 black balls, each drawing in order, and the whole sum is to be received by him who first draws a white ball. What are their expectations?
 - (i.) Replacing the balls.
 - (ii.) Not replacing the balls.

CHAPTER XXIV.

FORMULAS.

SIMPLE INTEREST.

447. *If in Interest*

The principal is represented by P ,
 interest on \$1 for one year by r ,
 amount of \$1 for one year by R ,
 number of years by n ,
 amount of P for n years by A .

Then $R = 1 + r$,
 Simple interest on P for a year $= Pr$,
 Amount of P for a year $= PR$,
 Simple interest on P for n years $= Pnr$,
 Amount of P for n years $= P(1 + nr)$.
 That is $A = P(1 + nr)$.

448. When any three of the quantities A , P , n , r are given, the fourth may be found.

Ex. Required the rate when \$500 in 4 years at simple interest amounts to \$610.

Here r is required, A , P , n are given.
 $A = P(1 + nr)$,
 or $A = P + Pnr$.

$$\therefore Pnr = A - P.$$

$$\therefore r = \frac{A - P}{Pn} = \frac{610 - 500}{2000} = 0.055.$$

$5\frac{1}{2}$ per cent. *Ans.*

449. Since P will in n years amount to A , it is evident that P at the present time may be considered *equivalent in value* to A due at the end of n years; so that P may be regarded as the *present worth* of a given future sum A .

Ex. Find the present worth of \$600, due in 2 years, the rate of interest being 6 per cent.

$$A = P(1 + nr).$$

$$\therefore P = \frac{A}{1 + nr} = \frac{\$600}{1 + 0.12} = \$535.71.$$

COMPOUND INTEREST.

450. When *compound* interest is reckoned payable *annually*,

The amount of P dollars in

$$\begin{aligned} 1 \text{ year is } & P(1 + r) = PR, \\ 2 \text{ years is } & PR(1 + r) = PR^2, \\ n \text{ years } & = PR^n. \end{aligned}$$

That is, $A = PR^n.$

Hence, also, $P = \frac{A}{R^n}.$

When compound interest is payable *semi-annually*,

The amount of P dollars in

$$\begin{aligned} \frac{1}{2} \text{ year } &= P\left(1 + \frac{r}{2}\right), \\ 1 \text{ year } &= P\left(1 + \frac{r}{2}\right)^2, \\ n \text{ years } &= P\left(1 + \frac{r}{2}\right)^{2n}. \end{aligned}$$

That is, $A = P\left(1 + \frac{r}{2}\right)^{2n}.$

When the interest is payable *quarterly*,

$$A = P\left(1 + \frac{r}{4}\right)^{4n}.$$

When the interest is payable *monthly*,

$$A = P \left(1 + \frac{r}{12} \right)^{12n}.$$

When interest is payable q times a year,

$$A = P \left(1 + \frac{r}{q} \right)^{qn}.$$

Ex. Find the present worth of \$500, due in 4 years, at 5 per cent compound interest.

$$A = P(1 + r)^4.$$

$$\therefore P = \frac{A}{(1 + r)^4} = \frac{\$500}{(1.05)^4} = \$411.36. \text{ Ans.}$$

SINKING FUNDS.

451. If the sum set apart at the end of each year to be put at compound interest is represented by S , then,

The sum at the end of the

$$\text{first year} = S,$$

$$\text{second year} = S + SR,$$

$$\text{third year} = S + SR + SR^2,$$

$$nth \text{ year} = S + SR + SR^2 + \dots + SR^{n-1}.$$

That is, the amount $A = S + SR + SR^2 + \dots + SR^{n-1}$.

$$\therefore AR = SR + SR^2 + SR^3 + \dots + SR^n.$$

$$\therefore AR - A = SR^n - S.$$

$$\therefore A = \frac{S(R^n - 1)}{R - 1},$$

or,

$$A = \frac{S(R^n - 1)}{r}.$$

- (1) If \$10,000 be set apart annually, and put at 6 per cent compound interest for 10 years, what will be the amount?

$$A = \frac{S(R^n - 1)}{r} = \frac{\$10,000(1.06^{10} - 1)}{0.06}.$$

By logarithms the amount is found to be \$131,740 (*nearly*).

- (2) A county owes \$60,000. What sum must be set apart annually, as a sinking fund, to cancel the debt in 10 years, provided money is worth 6 per cent?

$$S = \frac{Ar}{R^n - 1} = \frac{\$60,000 \times 0.06}{1.06^{10} - 1} = \$4555 \text{ (nearly).}$$

NOTE. The amount of tax required yearly is \$3600 for the *interest* and \$4555 for the sinking fund; that is, \$8155.

ANNUITIES.

452. A sum of money that is payable yearly, or in parts at fixed periods in the year, is called an **annuity**.

I. *To find the amount of an unpaid annuity when the interest, time, and rate per cent are given.*

The sum due at the *end* of the

first year = S ,

second year = $S + SR$,

third year = $S + SR + SR^2$,

n th year = $S + SR + SR^2 + \dots + SR^{n-1}$.

That is, $A = \frac{S(R^n - 1)}{r}$. § 451.

Ex. An annuity of \$1200 was unpaid for 6 years. What was the amount due if interest is reckoned at 6 per cent?

$$A = \frac{S(R^n - 1)}{r} = \frac{\$1200(1.06^6 - 1)}{0.06} = \$8370.$$

II. *To find the present worth of an annuity when the time it is to continue and the rate per cent are given.*

Let P denote the present worth. Then the amount of P for n years will be equal to A , the amount of the annuity for n years.

But the amount of P for n years

$$= P(1 + r)^n = PR^n,$$

and $A = \frac{S(R^n - 1)}{R - 1}$. § 451.

$$\therefore PR^n = \frac{S(R^n - 1)}{R - 1}.$$

$$\therefore P = \frac{S}{R^n} \times \frac{R^n - 1}{R - 1}.$$

This equation may be written

$$P = \frac{S}{R - 1} \times \frac{R^n - 1}{R^n}.$$

If the annuity is *perpetual*, the fraction

$$\frac{R^n - 1}{R^n}$$

approaches to *unity* as its limit.

$$\therefore P = \text{limit of } \frac{S}{R - 1} \times \frac{R^n - 1}{R^n} = \frac{S}{R - 1} = \frac{S}{r}.$$

- (1) Find the present worth of an annual pension of \$105, for 5 years, at 4 per cent interest.

$$\begin{aligned} P &= \frac{S}{R^n} \times \frac{R^n - 1}{R - 1} \\ &= \frac{\$105}{1.04^5} \times \frac{1.04^5 - 1}{1.04 - 1} = \$467 \text{ (nearly)}. \end{aligned}$$

- (2) Find the present worth of a perpetual scholarship that pays \$300 annually, at 6 per cent interest.

$$P = \frac{S}{r} = \frac{\$300}{0.06} = \$5000.$$

III. To find the present worth of an annuity that begins in a given number of years, when the time it is to continue and the rate per cent are given.

Let p denote the number of years before the annuity begins, and q the number of years the annuity is to continue.

Then the present worth of the annuity to the time it *terminates* is

$$\frac{S}{R^{p+q}} \times \frac{R^{p+q} - 1}{R - 1},$$

and the present worth of the annuity to the time it *begins* is

$$\frac{S}{R^p} \times \frac{R^p - 1}{R - 1},$$

$$\text{Hence, } P = \left(\frac{S}{R^{p+q}} \times \frac{R^{p+q} - 1}{R - 1} \right) - \left(\frac{S}{R^p} \times \frac{R^p - 1}{R - 1} \right).$$

$$\therefore P = \frac{S}{R^{p+q}} \times \frac{R^q - 1}{R - 1}.$$

If the annuity is to begin at the end of p years, and to be perpetual, the formula

$$P = \frac{S}{R^{p+q}} \times \frac{R^q - 1}{R - 1}$$

becomes
$$P = \frac{S}{R^p(R - 1)} \times \frac{R^q - 1}{R^q}.$$

And since the limit of $\frac{R^q - 1}{R^q}$ is unity,

$$P = \text{the limit of } \frac{S}{R^p(R - 1)} \times \frac{R^q - 1}{R^q} = \frac{S}{R^p(R - 1)}.$$

- (1) Find the present worth of an annuity of \$5000, to begin in 6 years, and to continue 12 years, at 6 per cent interest.

$$\begin{aligned} P &= \frac{S}{R^{p+q}} \times \frac{R^q - 1}{R - 1} \\ &= \frac{\$5000}{1.06^{18}} \times \frac{1.06^{12} - 1}{0.06} = \$29,550. \end{aligned}$$

- (2) Find the present worth of a perpetual annuity of \$1000, to begin in 3 years, at 4 per cent interest.

$$P = \frac{S}{R^p(R - 1)} = \frac{\$1000}{1.04^3 \times 0.04} = \$22,225.$$

IV. *To find the annuity when the present worth, the time, and the rate per cent are given.*

$$\begin{aligned} P &= \frac{S(R^n - 1)}{R^n(R - 1)} \\ \therefore S &= \frac{PR^n(R - 1)}{R^n - 1} = Pr \times \frac{R^n}{R^n - 1}. \end{aligned}$$

- (1) What annuity for 5 years will \$4675 give when interest is reckoned at 4 per cent ?

$$S = Pr \times \frac{R^n}{R^n - 1} = \$4675 \times 0.04 \times \frac{1.04^5}{1.04^5 - 1} = \$1050.$$

LIFE INSURANCE.

453. In order that a certain sum may be secured, to be payable at the death of a person, he pays yearly a fixed *premium*.

If P denotes the premium to be paid for n years to insure an amount A , to be paid immediately after the last premium, then

$$A = \frac{P(R^n - 1)}{R - 1}. \quad \S\ 451.$$

$$\therefore P = \frac{A(R - 1)}{R^n - 1} = \frac{Ar}{R^n - 1}.$$

If A is to be paid a year after the last premium, then

$$P = \frac{A(R - 1)}{R(R^n - 1)} = \frac{Ar}{R(R^n - 1)}.$$

NOTE. In the calculation of life insurance it is necessary to employ tables which show for any age the probable duration of life.

BONDS.

454. If P denote the price of a bond that has n years to run, and bears r per cent interest, S the face of the bond, and q the current rate of interest, what interest on his investment will a purchaser of such a bond receive?

Let x denote the rate of interest on the investment.

Then $P(1+x)^n$ is the value of the purchase money at the end of n years.

$Sr(1+q)^{n-1} + Sr(1+q)^{n-2} + \dots + Sr + S$ is the amount of the money received on the bond if the interest received from the bond is put immediately at compound interest at q per cent.

But $Sr(1+q)^{n-1} + Sr(1+q)^{n-2} + \dots + Sr + S = S + \frac{Sr[(1+q)^n - 1]}{q}$.

$$\therefore P(1+x)^n = S + \frac{Sr[(1+q)^n - 1]}{q}.$$

$$\begin{aligned} \therefore 1+x &= \left(\frac{S}{P} + \frac{Sr[(1+q)^n - 1]}{Pq} \right)^{\frac{1}{n}} \\ &= \left(\frac{Sq + Sr(1+q)^n - Sr}{Pq} \right)^{\frac{1}{n}}. \end{aligned}$$

- (1) What interest does a person receive on his investment if he buys a 4 per cent bond, at 114, that has 26 years to run, and if money is worth $3\frac{1}{2}$ per cent?

$$1 + x = \left(\frac{3.5 + 4(1.035)^{26} - 4}{3.99} \right)^{\frac{1}{26}}.$$

By logarithms, $1 + x = 1.033$.

That is, the purchaser will receive $3\frac{1}{2}$ per cent for his money.

- (2) At what price must 7 per cent bonds be bought, running 12 years, with the interest payable semi-annually, in order that the purchaser may receive on his investment 5 per cent, interest semi-annual, which is the current rate of interest?

$$P(1+x)^n = \frac{Sq + Sr(1+q)^n - Sr}{q}.$$

$$\therefore P = \frac{Sq + Sr(1+q)^n - Sr}{q(1+x)^n}.$$

In this case $S = 100$; and, as the interest is semi-annual,

$$q = 0.025, r = 0.035, n = 24, x = 0.025.$$

$$\text{Hence, } P = \frac{2.5 + 3.5(1.025)^{24} - 3.5}{0.025(1.025)^{24}}.$$

By logarithms, $P = 118$.

EXERCISE 124.

1. In how many years will \$100 amount to \$1050, at 5 per cent compound interest?
2. In how many years will \$A amount to \$B (i.) at simple interest, (ii.) at compound interest, r and R being used in their usual sense?
3. Find the difference (to five places of decimals) between the amount of \$1 in 2 years, at 6 per cent compound interest, according as the interest is due yearly or monthly.
4. At 5 per cent, find the amount of an annuity A which has been unpaid for 4 years.

5. Find the present value of an annuity of \$100 for 5 years, reckoning interest at 4 per cent.
6. A perpetual annuity of \$1000 is to be purchased, to begin at the end of 10 years. If interest is reckoned at $3\frac{1}{2}$ per cent, what should be paid for it?
7. A debt of \$1850 is discharged by two payments of \$1000 each, at the end of one and two years. Find the rate of interest paid.
8. Reckoning interest at 4 per cent, what annual premium should be paid for 30 years, in order to secure \$2000 to be paid at the end of that time, the premium being due at the beginning of each year?
9. An annual premium of \$150 is paid to a life-insurance company for insuring \$5000. If money is worth 4 per cent, for how many years must the premium be paid in order that the company may sustain no loss?
10. What may be paid for bonds due in 10 years, and bearing semi-annual coupons of 4 per cent each, in order to realize 3 per cent semi-annually, if money is worth 3 per cent semi-annually?
11. When money is worth 2 per cent semi-annually, if bonds having 12 years to run and bearing semi-annual coupons of $3\frac{1}{2}$ per cent each, are bought at 114 $\frac{1}{8}$, what per cent is realized on the investment?
12. If \$126 is paid for bonds due in 12 years, and yielding $3\frac{1}{2}$ per cent semi-annually, what per cent is realized on the investment, provided money is worth 2 per cent semi-annually?
13. A person borrows \$600.25. How much must he pay annually that the whole debt may be discharged in 35 years, allowing simple interest at 4 per cent?

14. A perpetual annuity of \$100 a year is sold for \$2500. At what rate is the interest reckoned?
15. A perpetual annuity of \$320, to begin 10 years hence, is to be purchased. If interest is reckoned at $3\frac{1}{2}$ per cent, what should be paid for it?
16. A sum of \$10,000 is loaned at 4 per cent. At the end of the first year a payment of \$400 is made; and at the end of each following year a payment is made greater by 30 per cent than the preceding payment. Find in how many years the debt will be paid.
17. A man with a capital of \$100,000 spends every year \$9000. If the current rate of interest is 5 per cent, in how many years will he be ruined?
18. Find the amount of \$365 at compound interest for 20 years at 5 per cent.
19. In how many years will \$20 amount to \$150, at 4 per cent compound interest?
20. At what rate per cent, compound interest, will \$2500 amount to \$3450 in 7 years?
21. If the population of a state increases in 10 years from 2,009,000 to 2,487,000, find the yearly rate of increase.
22. The population of a State now is 1,918,600, and the yearly rate of increase is 2.38 per cent. Determine its population 10 years hence.
23. A banker borrows a sum of money at $3\frac{1}{2}$ per cent, interest payable annually, and loans the same at 5 per cent, interest payable quarterly. If his annual gain is \$441, determine the sum borrowed.

CHAPTER XXV.

CONTINUED FRACTIONS.

455. A FRACTION in the form of

$$\frac{a}{b + \frac{c}{d + \frac{e}{f} + \text{etc.}}}$$

is called a **Continued Fraction**, though the term is commonly restricted to a continued fraction that has 1 for each of its numerators, as

$$\frac{1}{p + \frac{1}{q + \frac{1}{r + \text{etc.}}}}$$

456. *Any proper fraction in its lowest terms may be converted into a terminated continued fraction.*

Let $\frac{b}{a}$ be such a fraction ;

then
$$\frac{b}{a} = \frac{1}{\frac{a}{b}} = \frac{1}{p + \frac{c}{b}}$$

(if p is the quotient and c the remainder of $a \div b$) ;

and

$$\frac{1}{p + \frac{c}{b}} = \frac{1}{p + \frac{1}{\frac{b}{c}}} = \frac{1}{p + \frac{1}{q + \frac{d}{c}}}$$

(if q is the quotient and d the remainder of $b \div c$),

$$= \frac{1}{p + \frac{1}{q + \frac{1}{r + \text{etc.}}}}$$

The successive steps of the process are the same as the steps for finding the G.C.M. of a and b ; and since a and b are prime to each other, a remainder, 1, will at length be reached, and the fraction terminates.

457. The fractions formed by taking one, two, three, of the quotients p, q, r, \dots , are

$$\frac{1}{p}, \frac{1}{p + \frac{1}{q}}, \frac{1}{p + \frac{1}{q + \frac{1}{r}}}, \dots,$$

which simplified are

$$\frac{1}{p}, \frac{q}{pq + 1}, \frac{qr + 1}{(pq + 1)r + p}, \dots,$$

and are called the **first, second, and third convergents**, respectively.

458. *The successive convergents are alternately greater and less than the true value of the given fraction.*

Let x be the true value of $\frac{1}{p + \frac{1}{q + \frac{1}{r + \text{etc.}}}}$

then, since p, q, r, \dots are *positive integers*,

$$p < p + \frac{1}{q + \frac{1}{r + \text{etc.}}}$$

$$\therefore \frac{1}{p} > \frac{1}{p + \frac{1}{q + \frac{1}{r + \text{etc.}}}}; \text{ that is, } \frac{1}{p} > x.$$

Again, $q < q + \frac{1}{r + \text{etc.}}$

$$\therefore \frac{1}{q} > \frac{1}{q + \frac{1}{r + \text{etc.}}}$$

$$\therefore \frac{1}{p + \frac{1}{q}} < \frac{1}{p + \frac{1}{q + \frac{1}{r + \text{etc.}}}}; \text{ that is, } \frac{1}{p + \frac{1}{q}} < x;$$

and so on.

459. If $\frac{u_1}{v_1}, \frac{u_2}{v_2}, \frac{u_3}{v_3}$ are any three consecutive convergents, and if m_1, m_2, m_3 are the quotients that produce them, then

$$\frac{u_3}{v_3} = \frac{m_3 u_2 + u_1}{m_3 v_2 + v_1}.$$

For, if the first three quotients are p, q, r , the first three convergents are

$$\frac{1}{p}, \frac{1}{p + \frac{1}{q}}, \frac{1}{p + \frac{1}{q + \frac{1}{r}}}, \quad (\text{i.})$$

that is,
$$\frac{1}{p}, \frac{q}{pq + 1}, \frac{qr + 1}{(pq + 1)r + p}. \quad (\text{ii.})$$

From (i.) it is seen that the second convergent is formed from the first by writing in it $p + \frac{1}{q}$ for p ; and the third from the second by writing $q + \frac{1}{r}$ for q . In this way, any convergent may be formed from the preceding convergent.

Therefore, $\frac{u_3}{v_3}$ will be formed from $\frac{u_2}{v_2}$ by writing $m_2 + \frac{1}{m_3}$ for m_2 .

In (ii.) it is seen that the third convergent has its numerator $= r \times (\text{second numerator}) + (\text{first numerator})$; and its denominator $= r \times (\text{second denominator}) + (\text{first denominator})$.

Assume that this law holds good for the *third* of the three consecutive convergents

$$\frac{u_0}{v_0}, \frac{u_1}{v_1}, \frac{u_2}{v_2}, \text{ so that, } \frac{u_2}{v_2} = \frac{m_2 u_1 + u_0}{m_2 v_1 + v_0};$$

then, since, by (ii.), $\frac{u_3}{v_3}$ is formed from $\frac{u_2}{v_2}$ by using $m_2 + \frac{1}{m_3}$ for m_2 ,

$$\frac{u_3}{v_3} = \frac{\left(m_2 + \frac{1}{m_3}\right) u_1 + u_0}{\left(m_2 + \frac{1}{m_3}\right) v_1 + v_0} = \frac{m_3 (m_2 u_1 + u_0) + u_1}{m_3 (m_2 v_1 + v_0) + v_1}.$$

Substitute u_2 and v_2 for their values $m_2 u_1 + u_0$ and $m_2 v_1 + v_0$;

then
$$\frac{u_3}{v_3} = \frac{m_3 u_2 + u_1}{m_3 v_2 + v_1}.$$

Therefore, the law still holds good; and as it has been shown to be true for the *third* convergent, the law is general.

460. *The difference between two consecutive convergents*

$$\frac{u_1}{v_1} \text{ and } \frac{u_2}{v_2} \text{ is } \frac{1}{v_1 v_2}.$$

The difference between the first two convergents

$$\frac{1}{p} - \frac{q}{p' + 1} = \frac{1}{p(pq + 1)}.$$

Let the sign \sim mean "the difference between," and assume the proposition true for

$$\frac{u_0}{v_0} \text{ and } \frac{u_1}{v_1} \text{ so that } \frac{u_0}{v_0} \sim \frac{u_1}{v_1} = \frac{u_0 v_1 \sim u_1 v_0}{v_0 v_1} = \frac{1}{v_0 v_1};$$

then

$$\begin{aligned} \frac{u^2}{v^2} \sim \frac{u_1}{v_1} &= \frac{u_2 v_1 \sim u_1 v_2}{v_1 v_2}; \\ &= \frac{(m_2 u_1 + u_0) v_1 \sim u_1 (m_2 v_1 + v_0)}{v_1 v_2} \end{aligned}$$

(by substituting for u_2 and v_2 their values, $m_2 u_1 + u_0$ and $m_2 v_1 + v_0$),

$$\begin{aligned} &= \frac{u_0 v_1 \sim u_1 v_0}{v_1 v_2}; \\ &= \frac{1}{v_1 v_2} \text{ (by the assumption).} \end{aligned}$$

Hence, if the proposition be true for one pair of consecutive convergents, it will be true for the next pair; but it has been shown to be true for the *first* pair, therefore it is true for *every* pair.

Since the real value of x lies between two consecutive convergents, $\frac{u_1}{v_1}$ and $\frac{u_2}{v_2}$, $\frac{u_1}{v_1}$ will differ from x by a quantity less than $\frac{u_1}{v_1} \sim \frac{u_2}{v_2}$; that is, by a quantity $< \frac{1}{v_1 v_2}$; so that the error in taking $\frac{u_1}{v_1}$ for x is $< \frac{1}{v_1 v_2}$, and therefore $< \frac{1}{v_1^2}$.

Any convergent, $\frac{u_1}{v_1}$, is in its lowest terms; for, if u_1 and v_1 had any common factor, it would also be a factor of $u_1 v_2 \sim u_2 v_1$ (§ 145); that is, a factor of 1.

461. *The successive convergents approach more and more nearly to the true value of the continued fraction.*

Let $\frac{u_0}{v_0}, \frac{u_1}{v_1}, \frac{u_2}{v_2}$ be consecutive convergents.

Now $\frac{u_2}{v_2}$ differs from x , the true value of the fraction, only by using m_2 instead of $m_2 + \frac{1}{m_3 + \text{etc.}}$.

Let this complete quotient, which is always greater than unity, be represented by M .

Then, since $\frac{u_2}{v_2} = \frac{m_2 u_1 + u_0}{m_2 v_1 + v_0}$, $x = \frac{M u_1 + u_0}{M v_1 + v_0}$.

$$\therefore x \sim \frac{u_1}{v_1} = \frac{M u_1 + u_0}{M v_1 + v_0} \sim \frac{u_1}{v_1} = \frac{u_0 v_1 \sim u_1 v_0}{v_1 (M v_1 + v_0)} = \frac{1}{v_1 (M v_1 + v_0)}.$$

$$\text{And } \frac{u_0}{v_0} \sim x = \frac{u_0}{v_0} \sim \frac{M u_1 + u_0}{M v_1 + v_0} = \frac{M (u_0 v_1 \sim u_1 v_0)}{v_0 (M v_1 + v_0)} = \frac{M}{v_0 (M v_1 + v_0)}.$$

Now $1 < M$ and $v_1 > v_0$, and for both these reasons:

$$x \sim \frac{u_1}{v_1} < \frac{u_0}{v_0} \sim x.$$

That is, $\frac{u_1}{v_1}$ is nearer to x than $\frac{u_0}{v_0}$ is.

462. *Any convergent $\frac{u_1}{v_1}$ is nearer the true value x than any other fraction with smaller denominator.*

Let $\frac{a}{b}$ be a fraction in which $b < v_1$.

If $\frac{a}{b}$ is one of the convergents, $x \sim \frac{a}{b} < \frac{u_1}{v_1} \sim x$. § 461.

If $\frac{a}{b}$ is not one of the convergents, and is nearer to x than $\frac{u_1}{v_1}$ is, then, since x lies between $\frac{u_1}{v_1}$ and $\frac{u_2}{v_2}$, $\frac{a}{b}$ must be nearer to $\frac{u_2}{v_2}$ than $\frac{u_1}{v_1}$ is; that is,

$$\frac{a}{b} \sim \frac{u_2}{v_2} < \frac{u_1}{v_1} \sim \frac{u_2}{v_2}, \text{ or } \frac{v_2 a \sim u_2 b}{v_2 b} < \frac{1}{v_1 v_2};$$

and since $b < v_1$, this would require $v_2 a \sim u_2 b$ to be < 1 ; but $v_2 a \sim u_2 b$ cannot be less than 1, for a, b, u_2, v_2 are all integers.

APPLICATIONS.

Ex. Find the continued fraction equal to $3\frac{1}{8}$, and also the successive convergents.

By following the process of finding the G.C.M. of 31, 75, the successive quotients are found to be 2, 2, 2, 1, 1, 2. Hence the continued fraction is

$$\cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2}}}}}}$$

To find the successive convergents :

Write the successive quotients in line, $\frac{0}{1}$ under the first quotient, $\frac{1}{1}$ under the second quotient, and then multiply each term by the quotient above it and add the term to the left to obtain the corresponding term to the right. Thus,

$$\text{Quotients} = 2, 2, 2, 1, 1, 2.$$

$$\text{Convergents} = \frac{0}{1}, \frac{1}{1}, \frac{2}{1}, \frac{5}{12}, \frac{7}{17}, \frac{13}{18}.$$

NOTE. It is convenient to begin to reckon with $\frac{0}{1}$, but the next convergent, in this case $\frac{1}{1}$, is called the *first* convergent.

463. A quadratic surd may be expressed in the form of a *non-terminating* continued fraction.

Ex. To express $\sqrt{3}$ in the form of a continued fraction.

$$\text{Suppose} \quad \sqrt{3} = 1 + \frac{1}{x} \text{ (for 1 is the greatest integer in } \sqrt{3}\text{),}$$

$$\text{then} \quad \frac{1}{x} = \sqrt{3} - 1.$$

$$\therefore x = \frac{1}{\sqrt{3}-1} = \frac{\sqrt{3}+1}{2} \text{ (by § 278).}$$

Suppose $\frac{\sqrt{3}+1}{2} = 1 + \frac{1}{y}$ (for 1 is the greatest integer in $\frac{\sqrt{3}+1}{2}$);

then
$$\frac{1}{y} = \frac{\sqrt{3}+1}{2} - 1 = \frac{\sqrt{3}-1}{2}.$$

$$\therefore y = \frac{2}{\sqrt{3}-1} = \frac{\sqrt{3}+1}{1} \text{ (by § 278).}$$

Suppose $\frac{\sqrt{3}+1}{1} = 2 + \frac{1}{z}$ (as 2 is the greatest integer in $\frac{\sqrt{3}+1}{1}$);

then
$$\frac{1}{z} = \frac{\sqrt{3}+1}{1} - 2 = \sqrt{3}-1.$$

$$\therefore z = \frac{1}{\sqrt{3}-1} \text{ (the same as } x \text{ above).}$$

$$\therefore \sqrt{3} = 1 + \frac{1}{1 + \frac{1}{2} + \text{etc.}}, \text{ of which } \frac{1}{1 + \frac{1}{2}} \text{ will be a continu-}$$

ally-repeating series, and the whole expression may be written,

$$1 + \frac{1}{1 + \frac{1}{2}}.$$

The convergents will be 1, 2, $\frac{3}{2}$, $\frac{5}{3}$, $\frac{8}{5}$, $\frac{13}{8}$, etc.

464. A continued fraction in which the denominators recur is called a **periodic** continued fraction.

465. The value of a periodic continued fraction can be expressed as the root of a quadratic equation.

Ex. Find the surd value of $\frac{1}{1 + \frac{1}{2}}$.

Let x be the value;

then
$$x = \frac{1}{1 + \frac{1}{2+x}};$$

that is,
$$x = \frac{2+x}{3+x},$$

whence
$$x = -1 + \sqrt{3}.$$

466. An exponential equation may be solved by continued fractions.

Ex. Solve $10^x = 2$, by continued fractions.

Suppose $x = 0 + \frac{1}{y}$,

then $10^{\frac{1}{y}} = 2$;

or $10 = 2^y$.

$\therefore y = 3 + \frac{1}{z}$ (as 10 lies between 2^3 and 2^4).

Then $10 = 2^3 \times 2^{\frac{1}{z}}$;

or $2^{\frac{1}{z}} = \frac{10}{8} = \frac{5}{4}$,

and $2 = (\frac{5}{4})^z$.

$\therefore z = 3 + \frac{1}{v}$.

Then $2 = (\frac{5}{4})^3 \times (\frac{5}{4})^{\frac{1}{v}}$;

or $(\frac{5}{4})^{\frac{1}{v}} = \frac{2}{1\frac{1}{8}}$,

and $\frac{5}{4} = (1\frac{1}{8})^v$.

The greatest integer in v will be found to be 9.

Hence,
$$x = 0 + \frac{1}{3 + \frac{1}{3 + \frac{1}{9} + \text{etc.}}}$$

The convergents will be $\frac{1}{3}$, $\frac{3}{10}$, $\frac{8}{33}$, etc.

$\therefore x = \frac{8}{33} = 0.3010$, nearly.

EXERCISE 125.

- Find continued fractions for $1\frac{1}{3}\frac{2}{7}$; $1\frac{1}{4}\frac{2}{7}$; $\sqrt{5}$; $\sqrt{11}$; $4\sqrt{6}$; and find the fifth convergent to each.
- Find continued fractions for $\frac{47}{257}$; $\frac{157}{864}$; $\frac{2065}{1188}$; $\frac{2991}{1688}$; and find the third convergent to each.
- Find continued fractions for $\sqrt{21}$; $\sqrt{22}$; $\sqrt{33}$; $\sqrt{55}$.
- Obtain convergents, with only two figures in the denominator, that approach nearest to the values of $\sqrt{10}$; $\sqrt{15}$; $\sqrt{17}$; $\sqrt{18}$; $\sqrt{20}$.

5. Find the proper fraction which, if converted into a continued fraction, will have quotients 1, 7, 5, 2.
6. Find the next convergent when the two preceding convergents are $\frac{3}{14}$ and $\frac{1}{8}$, and the next quotient is 5.
7. If the pound troy is the weight of 22.8157 inches of water, and the pound avoirdupois of 27.7274 inches, find a fraction with denominator < 100 which shall differ from their ratio by < 0.0001 .
8. The ratio of the diagonal to a side of a square being $\sqrt{2}$, find a fraction with denominator < 100 which shall differ from their ratio by < 0.0001 .
9. The ratio of the circumference of a circle to its diameter being 3.14159265, find the first three convergents, and determine to how many decimal places each may be depended upon as agreeing with the true value.
10. Two scales whose zero points coincide have the distances between consecutive divisions of the one to those of the other as 1 : 1.06577. Find what division-points most nearly coincide.
11. Find the surd values of

$$3 + \frac{1}{1 + \frac{1}{6}}; \quad \frac{1}{3 + \frac{1}{1 + \frac{1}{6}}}; \quad 1 + \frac{1}{2 + \frac{1}{3 + \frac{1}{4}}}.$$
12. Show that the ratio of the diagonal of a cube to its edge may be nearly expressed by 97 : 56. Find the limit of the error made in taking this ratio for the true ratio.
13. Find a series of fractions converging to the ratio of 5 hours 48 minutes 51 seconds to 24 hours.
14. Find a series of fractions converging to the ratio of a cubic yard to a cubic meter, if 1 cubic yard = 0.76453 of a cubic meter.

CHAPTER XXVI.

THEORY OF LIMITS.

467. WHEN a quantity is regarded as having a *fixed* value, it is called a **Constant**; but when it is regarded, under the conditions imposed upon it, as having an *indefinite number of different* values, it is called a **Variable**.

468. When it can be shown that the value of a variable, measured at a series of definite intervals, can, by indefinite continuation of the series, be made to differ from a given constant by less than any assigned quantity however small, but cannot be made absolutely equal to it, the constant is called the **Limit** of the variable; and the variable is said to *approach indefinitely to its limit*.

469. In order, then, for a fixed value to be the limit of a variable value, it is necessary and sufficient that there be *some* difference between the variable and the fixed value, but that this difference may be made *as small as we please*.

Consider the series

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \dots$$

The general formula for the sum of n terms of this series is (§ 395),

$$\frac{a}{1-r} - \frac{ar^n}{1-r}.$$

$$\text{When } n = 3, \text{ the sum equals } \frac{1}{1} - \frac{\frac{1}{8}}{1} = 2 - \frac{1}{4}.$$

$$\text{When } n = 4, \text{ the sum equals } \frac{1}{1} - \frac{\frac{1}{16}}{1} = 2 - \frac{1}{8}.$$

When $n = 5$, the sum equals $\frac{1}{\frac{1}{2}} - \frac{\frac{1}{2}}{\frac{1}{2}} = 2 - \frac{1}{16}$.

When $n = 6$, the sum equals $\frac{1}{\frac{1}{2}} - \frac{\frac{1}{4}}{\frac{1}{2}} = 2 - \frac{1}{32}$.

As n increases, the sum approaches 2 as a limit; for, however great n becomes, there will continue to be some difference between the variable and 2, and this difference, by increasing n , may be indefinitely diminished.

470. In the above series the variable sum is increasing towards its limit, and each successive value is less than its limit.

A variable may decrease towards its limit, and each successive value be greater than its limit.

A variable, in approaching its limit, may be sometimes greater and sometimes less than its limit.

Thus, in the series

$$1, -\frac{1}{2}, +\frac{1}{4}, -\frac{1}{8}, +\frac{1}{16}, -\frac{1}{32}, + \dots,$$

for which a of the general formula, § 395, is 1, and r is $-\frac{1}{2}$.

When $n = 3$, the sum equals $\frac{1}{\frac{1}{2}} - \frac{-\frac{1}{2}}{\frac{1}{2}} = \frac{2}{3} + \frac{1}{12}$.

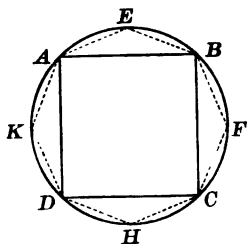
When $n = 4$, the sum equals $\frac{1}{\frac{1}{2}} - \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{2}{3} - \frac{1}{24}$.

When $n = 5$, the sum equals $\frac{1}{\frac{1}{2}} - \frac{-\frac{1}{8}}{\frac{1}{2}} = \frac{2}{3} + \frac{1}{48}$.

When $n = 6$, the sum equals $\frac{1}{\frac{1}{2}} - \frac{\frac{1}{16}}{\frac{1}{2}} = \frac{2}{3} - \frac{1}{96}$.

As n increases, the sum evidently approaches $\frac{2}{3}$ as a limit; and for every *odd* number of terms is *greater*, for every *even* number of terms is *less*, than the limit. Each successive value of the variable differs from $\frac{2}{3}$ less than the preceding value; and though the successive values of the variable are alternately greater and less than the limit, the difference between the variable and the limit diminishes indefinitely by increasing n .

471. As a geometrical representation of the indefinite approach of a variable to its limit, suppose a square $ABCD$ inscribed in a circle $ABCD$.



Connect the extremities of each side of the square with points of the circumference equally distant from them, namely, the points E, F, H, K .

The length of the perimeter represented by the dotted lines is greater than that of the square, since two sides replace each side of the square and form with it a triangle; and two sides of a triangle are together greater than the third side.

By continually repeating the process of doubling the number of sides of each resulting inscribed figure, the length of the perimeter will increase with the increase of the number of sides; but it cannot become equal to the length of the circumference, for the perimeter will continue to be made up of straight lines, each one of which is less than the part of the circumference between its extremities.

The length of the circumference is therefore the limit of the length of the perimeter as the number of sides of the inscribed figure is indefinitely increased.

THEOREMS OF LIMITS.

472. *If two variables are equal and are so related that a change in one produces such a change in the other that they continue equal, and each approaches a limit, their limits are equal.*

Let x and y represent two equal variables which increase towards their respective limits a and b , and which continue equal in approaching their limits; then $a = b$.

If a and b are not equal, one, as a , is the greater.

Let $d = a - b$.

Since x approaches a indefinitely, the difference between a and x may be made less than d .

Then $a - x < d$;
that is, $a - x < a - b$.
Therefore, $x > b$.

Since b is the limit of y , and y increasing,

$y < b$.
But $x > b$.
Therefore, $y < x$.

But this is contrary to the supposition that x and y are equal.

Therefore, a and b cannot be unequal. Hence, $a = b$.

When the variables are decreasing towards their limits, if a and b are unequal, one, as b , is the greater.

Let $d = b - a$.

Since x approaches a indefinitely, the difference between x and a may be made less than d .

Then $x - a < d$;
that is, $x - a < b - a$.
Therefore, $x < b$.

Since b is the limit of y , and y decreasing,

$y > b$.
But $x < b$.
Therefore, $y > x$.

But this is contrary to the supposition that x and y are equal.

Therefore, a and b cannot be unequal. Hence, $a = b$.

473. *If two variables have a fixed ratio and are so related that a change in one produces such a change in the other that they continue to have this ratio, and each approaches a limit, their limits are in the same ratio.*

Let x and y represent the two variables, r their ratio, a and b their respective limits.

Then $x : y = r : 1$; that is, $x = r \times y$.

By § 472, the limit of $x = r \times$ (the limit of y);
that is, $a = r \times b$; hence, $a : b = r : 1$.

474. *The limit of the sum of two or more variables is equal to the sum of their respective limits.*

Let x, y, z, \dots represent the variables.

a, b, c, \dots represent their respective limits,

and d, d', d'', \dots represent the differences between the variables and their respective limits.

If the variables are increasing,

$$x = a - d,$$

$$y = b - d',$$

$$z = c - d''; \text{ and so on.}$$

$$\therefore x + y + z + \dots = a + b + c + \dots - (d + d' + d'' + \dots).$$

It is required to show that $d + d' + d'' + \dots$ can be made less than any assigned quantity.

Let q represent any assigned quantity; let $d, d',$ and d'', \dots be n in number, and d the greatest of them. Then

$$d + d' + d'' + \dots < nd.$$

Now, since d , the difference between x and its limit a , may be diminished at pleasure, it may be made less than $\frac{q}{n}$, so that,

$$d < \frac{q}{n}, \text{ and therefore } nd < q.$$

$$\text{But } d + d' + d'' + \dots < nd.$$

$$\text{Therefore, } d + d' + d'' + \dots < q;$$

that is the difference between the sum of the variables $x + y + z + \dots$ and the sum of their respective limits $a + b + c + \dots$, can be made less than any assigned quantity.

Therefore, the limit of $x + y + z + \dots = a + b + c + \dots$

475. *The limit of the product of two variables is equal to the product of their limits.*

Let x and y represent two variables, a and b their respective limits, d and d' the differences between the variables and their limits.

If the variables are increasing,

$$x = a - d,$$

$$y = b - d'.$$

$$\begin{aligned} \therefore xy &= ab - db - d'a + dd' \\ &= ab - (db + d'a - dd'). \end{aligned}$$

Since d and d' may be made as small as we please, and a and b are finite values, the value of the expression $db + d'a - dd'$ may be made as small as we please; that is, the difference between the product of the two variables and the product of their limits can be made less than any assigned quantity.

Therefore, the limit of $xy = ab$.

If there are three or more variables, the proposition may be proved for two variables, then for this product and a third variable, and so on.

476. By considering the variables x, y, z, \dots all equal, and the limits a, b, c, \dots equal, it follows that,

The limit of any power of a variable is equal to that power of its limit.

477. *The limit of the quotient of two variables is equal to the quotient of their limits.*

Let x and y represent the variables,

a and b represent their respective limits,

d and d' represent the differences between the variables and their limits.

If the variables are increasing,

$$x = a - d \text{ and } y = b - d'.$$

Therefore, $\frac{a-d}{b-d'}$ is the quotient of the variables,

also $\frac{a}{b}$ is the quotient of the limits,

and $\frac{a}{b} - \frac{a-d}{b-d'}$ is the difference of the quotients.

$$\text{But } \frac{a}{b} - \frac{a-d}{b-d'} = -\frac{ad' - bd}{b(b-d')},$$

and since d and d' may be decreased at pleasure, the dividend $ad' - bd$ may be made less than any assigned value.

Therefore, if b is not zero, this quotient can be made less than any assigned value.

Hence, the limit of $\frac{a-d}{b-d'} = \frac{a}{b}$; that is, the limit of $\frac{x}{y} = \frac{a}{b}$.

478. To find the limit of the quotient $\frac{x^n - a^n}{x - a}$, when x approaches a as a limit.

By division, $\frac{x^n - a^n}{x - a} = x^{n-1} + ax^{n-2} + a^2x^{n-3} + \dots a^{n-1}$.

As x approaches a , x^{n-1} approaches a^{n-1} ,
 ax^{n-2} approaches a^{n-1} ,
 a^2x^{n-3} approaches a^{n-1} ; and so on.

Since there are n terms in the right member, the limit of

$$\frac{x^n - a^n}{x - a} = na^{n-1} \text{ as } x \text{ approaches } a.$$

This result may be shown to be true whether n is integral or fractional, positive or negative.

CONVERGENCY OF INFINITE SERIES.

479. A convergent series is a series whose sum, as the number of its terms is indefinitely increased, approaches some fixed finite value as a limit. § 369.

480. The sum of n terms of the geometrical progression a, ar, ar^2, \dots has been shown to be

$$a \left(\frac{r^n - 1}{r - 1} \right). \quad \text{§ 391.}$$

When $r > 1$, r^n increases with the increase of n , and the sum of the series approaches no fixed limit. Hence, in this case the series is not convergent.

When $r = 1$, the sum equals $\frac{0}{0}$, which represents no definite value. Hence, in this case the series is not convergent.

When $r = -1$, the sum equals 0 or a , according as n is an even or an odd number. Hence, in this case the sum of the series does not *indefinitely approach* to a fixed limit by increasing n , but fluctuates between 0 and a , and the series is not convergent.

When $r < 1$, r^n , with the increase of n , approaches 0 as a limit, and the sum approaches the fixed value $\frac{a}{1-r}$. Hence, in this case the series is convergent.

481. It is necessary and sufficient for the convergency of an infinite series that the *sum of the remaining terms after the n th* should approach 0 as a limit, as n increases indefinitely.

482. If, in a series of positive terms, the sum of the remaining terms after the n th approaches 0 as a limit, it is evident that each separate term after the n th must approach 0 as a limit.

483. If each term after the n th approaches 0 as a limit, it does not necessarily follow that the *sum* of all the terms after the n th approaches 0 as a limit.

(1) To determine whether the sum of the Harmonical series is convergent :

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, \frac{1}{n}, \frac{1}{n+1}, \frac{1}{n+2}, \dots$$

Each term after the n th approaches 0 as n increases.

The sum of n terms after the n th term is

$$\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \dots + \frac{1}{2n},$$

which is greater than $\frac{1}{2n} + \frac{1}{2n} + \dots$ to n terms; and therefore greater than $n \times \frac{1}{2n}$; that is, greater than $\frac{1}{2}$.

Now, the first term is 1, the second term is $\frac{1}{2}$, the sum of the next two terms is greater than $\frac{1}{2}$, the sum of the succeeding four terms is greater than $\frac{1}{2}$; and so on. So that, by increasing n , the sum will approach a number greater than any finite multiple of $\frac{1}{2}$.

Therefore, the series is divergent.

(2) To determine whether the following series is convergent :

$$1 + \frac{1}{1} + \frac{1}{1 \times 2} + \frac{1}{1 \times 2 \times 3} + \dots + \frac{1}{n-1} + \frac{1}{n} + \frac{1}{n+1} + \dots$$

The n th term of this series is evidently $\frac{1}{n-1}$.

The sum of the remaining terms is $\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \dots$

which is $\frac{1}{n} \left(1 + \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \dots \right)$,

and this is $< \frac{1}{n} \left(1 + \frac{1}{n} + \frac{1}{n^2} + \dots \right)$

therefore, $< \frac{1}{n} \left(\frac{1}{1 - \frac{1}{n}} \right)$, for, $\frac{1}{1 - \frac{1}{n}}$ expanded $= 1 + \frac{1}{n} + \frac{1}{n^2} + \dots$

therefore, $< \frac{1}{n} \left(\frac{n}{n-1} \right)$,

therefore, $< \frac{n}{n} \left(\frac{1}{n-1} \right)$,

therefore, $< \frac{1}{(n-1) \frac{n-1}{n}}$.

But $\frac{1}{(n-1) \frac{n-1}{n}}$ approaches 0 as n increases.

Therefore, the sum of the series is convergent.

TESTS OF THE CONVERGENCY OF SERIES.

484. *If the terms of a series are all positive, and the limit of the n th term is 0, then if the limit of the ratio of the $(n+1)$ th term to the n th term is less than 1, the series is convergent.*

In the infinite series

$$a_1, a_2, a_3, \dots, a_n, a_{n+1}, a_{n+2}, \dots,$$

let q , after a certain value of n , represent the limit of the ratio $\frac{a_{n+1}}{a_n}$ and be less than 1.

Also, let k be some fixed value between q and 1. Then

$$\begin{aligned}\frac{a_{n+1}}{a_n} &< k, & \frac{a_{n+2}}{a_{n+1}} &< k, & \frac{a_{n+3}}{a_{n+2}} &< k, & \dots \\ \therefore a_{n+1} &< ka_n, & a_{n+2} &< k^2 a_n, & a_{n+3} &< k^3 a_n. \\ \therefore a_{n+1} + a_{n+2} + a_{n+3} + \dots &< a_n (k + k^2 + k^3 + \dots).\end{aligned}$$

Since k is less than 1, $\frac{k}{1-k} = k + k^2 + k^3 + \dots$

$$\therefore a_{n+1} + a_{n+2} + a_{n+3} + \dots < a_n \times \frac{k}{1-k}.$$

Since a_n has a certain finite value, $a_n \times \frac{k}{1-k}$ has a certain finite value.

Hence the infinite series beginning with a_{n+1} is convergent; and by adding to this the fixed finite sum $a_1 + a_2 + a_3 + \dots + a_n$, the whole series is convergent.

Thus, in the series

$$1 + \frac{1}{1} + \frac{1}{1 \times 2} + \frac{1}{1 \times 2 \times 3} + \dots + \frac{1}{1 \times 2 \times 3 \dots (n-1)} + \dots$$

$$\frac{a_{n+1}}{a_n} = \frac{1}{n}.$$

Hence, the ratio $\frac{a_{n+1}}{a_n}$ approaches zero with the increase of n , and the series is convergent; which agrees with what is shown in example (2), § 483.

If $q > 1$, there must be in the series some term from which the succeeding term is greater than the next preceding term; so that the remaining terms will form an increasing series, and therefore the series is not convergent.

If $q = 1$, this value gives no explanation as to whether the series is convergent or not; and in such cases other tests must be applied.

If q approaches 1 indefinitely as a limit, then no fixed value k can be found which will always lie between q and 1, and other tests of convergence must be applied.

Thus, in the infinite series

$$\frac{1}{1^r} + \frac{1}{2^r} + \frac{1}{3^r} + \dots + \frac{1}{n^r} + \frac{1}{(n+1)^r} + \dots,$$

q , the ratio of the $(n+1)$ th term to the n th term, equals

$$\left(\frac{n}{n+1}\right)^r = \left(1 - \frac{1}{n+1}\right)^r, \text{ which approaches 1 as } n \text{ increases.}$$

Suppose r positive and greater than 1; then the first term of the series is 1. The sum of the next two terms is less than $\frac{2}{2^r}$. The sum of the next four terms is less than $\frac{4}{4^r}$. The sum of the next eight terms is less than $\frac{8}{8^r}$; and so on.

Hence, the sum of the series is less than

$$1 + \frac{2}{2^r} + \frac{4}{4^r} + \frac{8}{8^r} + \dots \quad \text{or} \quad < 1 + \frac{1}{2^{r-1}} + \frac{1}{4^{r-1}} + \frac{1}{8^{r-1}} + \dots,$$

which is evidently convergent when r is positive and greater than 1.

If r is positive and equal to 1, the given series becomes

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots,$$

which is the Harmonical series, and is shown in § 483 to be divergent.

If r is negative, or less than 1, each term of the series is then greater than the corresponding term in the Harmonical series, and hence the series is divergent.

485. *If the terms of a series are alternately positive and negative, then when the terms continually decrease, and the limit of the n th term is zero, the series is convergent.*

Consider the infinite series,

$$a_1 - a_2 + a_3 - a_4 + \dots \mp a_n \pm a_{n+1} \mp a_{n+2} \pm \dots$$

The sum of the terms after the n th term is

$$\pm [a_{n+1} - (a_{n+2} - a_{n+3}) - (a_{n+4} - a_{n+5}) - \dots],$$

which may be written

$$\pm [a_{n+1} - a_{n+2} + (a_{n+3} - a_{n+4}) + (a_{n+5} - a_{n+6}) + \dots].$$

Since the terms are continually diminishing, the series of groups in either form of expression are positive, and therefore the absolute value of the required sum is seen, from the first form of expression, to be less than a_{n+1} ; and from the second form of expression, to be greater than $a_{n+1} - a_{n+2}$. But both a_{n+1} and a_{n+2} approach zero with the increase of n ; therefore the sum of the series after the n th term approaches zero, and the series is convergent.

It will be seen that if, in finding the sum of an infinite decreasing series of which the terms are alternately positive and negative, we stop at any term, the error will be less than the next succeeding term.

CHAPTER XXVII.

INDETERMINATE COEFFICIENTS.

486. A SERIES, $ax + bx^2 + cx^3 + dx^4 + \dots$, in which the coefficients, $a, b, c, d \dots$ are finite, may, by taking x sufficiently small, be made less than any assigned quantity.

For if q be any assigned quantity, and k the greatest of the coefficients a, b, c, \dots , then

$$ax + bx^2 + cx^3 + \dots < kx + kx^2 + kx^3 + \dots$$

But
$$kx + kx^2 + kx^3 + \dots = \frac{kx}{1-x},$$

(as is evident by dividing kx by $1-x$).

$$\therefore ax + bx^2 + cx^3 + \dots < \frac{kx}{1-x}, \text{ if } x \text{ be taken less than } 1.$$

Hence, if $\frac{kx}{1-x}$ be taken less than q .

that is, if $x < \frac{q}{q+k}$,

then $ax + bx^2 + cx^3 + \dots$ will be less than q .

THEOREM OF INDETERMINATE COEFFICIENTS.

487. If the two series,

$$A + Bx + Cx^2 + \dots$$

and
$$A' + B'x + C'x^2 + \dots$$

are equal for *all finite values* of x , then

$$A = A', \quad B = B', \quad C = C', \quad \text{and so on.}$$

For, since $A + Bx + Cx^2 + \dots = A' + B'x + C'x^2 + \dots$,
by transposition, $A - A' = (B' - B)x + (C' - C)x^2 + \dots$

Now, by taking x sufficiently small, the right side of this equation can be made *less* than any assigned value whatever (§ 486); and therefore less than $A - A'$, if $A - A'$ have any value whatever. Hence, $A - A'$ cannot have any value.

Therefore, $A - A' = 0$ or $A = A'$.

Hence, $Bx + Cx^2 + Dx^3 + \dots = B'x + C'x^2 + D'x^3 + \dots$
or $(B - B')x = (C' - C)x^2 + (D' - D)x^3 + \dots$;
by dividing by x ,

$$B - B' = (C' - C)x + (D' - D)x^2 + \dots;$$

and, by the same proof as for $A - A'$,

$$B - B' = 0 \text{ or } B = B'.$$

In like manner, $C = C'$, $D = D'$, and so on.

Hence, the equation

$$A + Bx + Cx^2 + \dots = A' + B'x + C'x^2 + \dots,$$

if true for all finite values of x , is an **identical equation**; that is, *the coefficients of like powers of x are the same*.

APPLICATIONS OF THE THEOREM.

(1) Expand $\frac{2+3x}{1+x+x^2}$ in ascending powers of x .

$$\text{Assume } \frac{2+3x}{1+x+x^2} = A + Bx + Cx^2 + Dx^3 + \dots,$$

then, by clearing of fractions,

$$\begin{aligned} 2 + 3x &= A + Bx + Cx^2 + Dx^3 + \dots \\ &\quad + Ax + Bx^2 + Cx^3 + \dots \\ &\quad + Ax^2 + Bx^3 + \dots; \end{aligned}$$

$$\therefore 2 + 3x = A + (B+A)x + (C+B+A)x^2 + (D+C+B)x^3 + \dots$$

Therefore, by the Theorem,

$$A = 2, \quad B + A = 3, \quad C + B + A = 0, \quad D + C + B = 0;$$

whence, $B = 1$, $C = -3$, $D = 2$, and so on.

$$\therefore \frac{2+3x}{1+x+x^2} = 2 + x - 3x^2 + 2x^3 - \dots$$

In employing the method of Indeterminate Coefficients, the form of the given expression must determine what powers of the variable x must be assumed. It is necessary and sufficient that the assumed equation, when simplified, shall have in the right member all the powers of x that are found in the left member.

If any powers of x occur in the right member that are not in the left member, the coefficients of these powers will be found to be 0, and they will vanish, so that in this case the process will not be vitiated; but if any powers of x occur in the left that are not in the right member, then the coefficients of these powers of x must be put equal to 0 in equating the coefficients of like powers of x ; and this leads to absurd results. Thus, if it were assumed in problem (1) that

$$\frac{2+3x}{1+x+x^2} = Ax + Bx^2 + Cx^3 + \dots$$

there would be in the equation simplified no term on the right corresponding to 2 on the left; so that, in equating the coefficients of like powers of x , 2, which is $2x^0$, would have to be put equal to $0x^0$; that is, $2 = 0$, a manifest absurdity.

(2) Expand $(a-x)^{\frac{1}{2}}$.

Assume $(a-x)^{\frac{1}{2}} = A + Bx + Cx^2 + Dx^3 + \dots$

Square $a-x = A^2 + 2ABx + (2AC + B^2)x^2 + (2AD + 2BC)x^3 + \dots$

Therefore, by the Theorem,

$$A^2 = a, 2AB = -1, 2AC + B^2 = 0, 2AD + 2BC = 0, \text{ etc.,}$$

and $A = a^{\frac{1}{2}}, B = -\frac{1}{2a^{\frac{1}{2}}}, C = -\frac{1}{8a^{\frac{3}{2}}}, D = -\frac{1}{16a^{\frac{5}{2}}}.$

$$\text{Hence, } (a-x)^{\frac{1}{2}} = a^{\frac{1}{2}} - \frac{x}{2a^{\frac{1}{2}}} - \frac{x^2}{8a^{\frac{3}{2}}} - \frac{x^3}{16a^{\frac{5}{2}}} - \dots$$

(3) Find the fraction in the form of $\frac{a+px}{p+qx+rx^2}$ which will produce, by executing the indicated division, the series $1-5x+6x^2+8x^3-40x^4-\dots$

Assume $\frac{1+Bx}{1+Cx+Dx^2}$ to be the fraction required.

then $\frac{1+Bx}{1+Cx+Dx^2} = 1-5x+6x^2+8x^3-\dots$

and, by clearing of fractions,

$$\begin{aligned} 1 + Bx &= 1 - 5x + 6x^2 + 8x^3 - \dots \\ &\quad + Cx - 5Cx^2 + 6Cx^3 + \dots \\ &\quad + Dx^2 - 5Dx^3 + \dots \end{aligned} \quad (i.)$$

$$\therefore B = -5 + C, \quad 6 - 5C + D = 0, \quad 8 + 6C - 5D = 0;$$

whence $D = 4$, $C = 2$, and $B = -3$.

$$\therefore \text{the fraction is } \frac{1 - 3x}{1 + 2x + 4x^2}.$$

NOTE. It will be seen in the product (i.) that the column which contains x^2 may be obtained by multiplying the *third* term of the *given series* by 1, the *second* term by Cx , and the *first* term by Dx^2 ; also, that the column which contains x^3 may be obtained by multiplying the *fourth* term of the given series by 1, the *third* term by Cx , and the *second* term by Dx^2 ; and that each column is equal to 0.

A series, any given number of whose consecutive terms are thus related, is called a **recurring series**; and the expression, $1 + Cx + Dx^2$, is called its **scale of relation**.

(4) If $y = ax + bx^2 + cx^3 + \dots$ find x in terms of y .

$$\text{Assume} \quad x = Ay + By^2 + Cy^3 + \dots;$$

But since $y = ax + bx^2 + cx^3 + \dots$, by substituting this value of y in the equation $x = Ay + By^2 + Cy^3 + \dots$

$$\begin{aligned} \text{the result is} \quad x &= Aax + Abx^2 + Acx^3 + \dots \\ &\quad + Ba^2x^2 + 2Babx^3 + \dots \\ &\quad + Ca^3x^3 + \dots \end{aligned}$$

$$\therefore Aa = 1, \quad Ab + Ba^2 = 0, \quad Ac + 2Bab + Ca^3 = 0;$$

$$\text{whence,} \quad A = \frac{1}{a}, \quad B = -\frac{b}{a^3}, \quad C = \frac{2b^2 - ac}{a^5}.$$

$$\therefore x = \frac{y}{a} - \frac{by^2}{a^3} + \frac{(2b^2 - ac)y^3}{a^5} - \dots$$

NOTE. Ex. (4) is an instance of **reversion of series**. If x in the given series is equal to 0, y will be equal to 0, and therefore no term in the required series will be clear of y . If, however, the given series is in the form of $y = a + bx + cx^2 + \dots$, it is necessary to substitute z for $y - a$, so that $z = bx + cx^2 + \dots$, and to express x in terms of z ; then to put $y - a$ in place of z .

If the given series $ax + bx^2 + cx^3 + \dots$ is an infinite series, then x must be less than 1, or the series will not be convergent; also, y , the sum of the given series, must be less than 1, or the assumed series $Ay + By^2 + Cy^3 + \dots$ will not be convergent.

- (5) Resolve $\frac{3x-7}{(x-2)(x-3)}$ into partial fractions.

The denominators will be $x-2$ and $x-3$.

Assume
$$\frac{3x-7}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3};$$

then
$$3x-7 = A(x-3) + B(x-2).$$

$$\therefore A+B=3 \text{ and } 3A+2B=7;$$

whence,
$$A=1 \text{ and } B=2.$$

Therefore,
$$\frac{3x-7}{(x-2)(x-3)} = \frac{1}{x-2} + \frac{2}{x-3}.$$

- (6) Resolve $\frac{3}{x^3+1}$ into partial fractions.

Since $x^3+1 = (x+1)(x^2-x+1)$, the denominators will be $x+1$ and x^2-x+1 .

Assume
$$\frac{3}{x^3+1} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}.$$

then
$$3 = A(x^2-x+1) + (Bx+C)(x+1)$$

$$= (A+B)x^2 + (B+C-A)x + (A+C);$$

whence,
$$3 = A+C, \quad B+C-A=0, \quad A+B=0;$$

and
$$A=1, \quad B=-1, \quad C=2.$$

Therefore,
$$\frac{3}{x^3+1} = \frac{1}{x+1} - \frac{x-2}{x^2-x+1}.$$

- (7) Resolve $\frac{4x^3-x^2-3x-2}{x^2(x+1)^2}$ into partial fractions.

The denominators may be $x, x^2, x+1, (x+1)^2$.

Assume
$$\frac{4x^3-x^2-3x-2}{x^2(x+1)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2}.$$

$$\therefore 4x^3-x^2-3x-2 = Ax(x+1)^2 + B(x+1)^2 + Cx^2(x+1) + Dx^2$$

$$= (A+C)x^3 + (2A+B+C+D)x^2 + (A+2B)x + B$$

whence,
$$A+C=4,$$

$$2A+B+C+D=-1,$$

$$A+2B=-3,$$

$$B=-2;$$

or
$$B=-2, \quad A=1, \quad C=3, \quad D=-4.$$

Therefore,
$$\frac{4x^3-x^2-3x-2}{x^2(x+1)^2} = \frac{1}{x} - \frac{2}{x^2} + \frac{3}{x+1} - \frac{4}{(x+1)^2}.$$

In decomposing a given fraction into its simplest partial fractions, it is important to determine what form the assumed fractions must have.

Since the given fraction is the *sum* of the required partial fractions, each assumed denominator must be a factor of the given denominator; moreover, all the factors of the given denominator must be taken as denominators of the assumed fractions.

Thus, if the given denominator can be expressed by

$$x^n (x \pm a)^n (x^2 + b)^n (x^2 \pm ax + b)^n,$$

the denominators of the assumed fractions must be

$$\begin{aligned} &x, x^2, \dots; \\ &(x \pm a), (x \pm a)^2, \dots; \\ &(x^2 + b), (x^2 + b)^2, \dots; \\ &(x \pm ax + b), (x^2 \pm ax + b)^2, \dots \end{aligned}$$

Since the required partial fractions are to be in their simplest form incapable of further decomposition, the numerator of each required fraction must be assumed with reference to this condition.

Thus, if the denominator is x^n or $(x \pm a)^n$, the assumed fraction must be of the form $\frac{A}{x^n}$ or $\frac{A}{(x \pm a)^n}$; for, if it had the form $\frac{Ax + B}{x^n}$ or $\frac{Ax + B}{(x \pm a)^n}$, it could be decomposed into two fractions, and the partial fractions would not be in the simplest form possible.

When all the monomial factors, and all the binomial factors, of the form $x \pm a$, have been removed from the denominator of the given expression, and there remains as a factor $(x^2 \pm ax + b)^n$ or $(x^2 + b)^n$, this is a quadratic which cannot be further resolved; and the numerator, therefore, *may* contain the first power of x , so that the assumed fraction must have the form

$$\frac{Ax + B}{x^2 \pm ax + b} \quad \text{or} \quad \frac{Ax + B}{x^2 + b}.$$

EXERCISE 126.

Expand to four terms in ascending powers of x :

1. $\frac{1}{2-3x}$ 2. $\frac{1+x}{2+3x}$ 3. $\frac{3-2x}{4-3x}$ 4. $\frac{1-x}{1-x+x^2}$
 5. $\frac{1}{1-2x+3x^2}$ 6. $\frac{5-2x}{1+3x-x^2}$ 7. $\frac{4x-6x^2}{1-2x+3x^2}$

Revert the series:

8. $y=x+x^2+x^3+\dots$ 10. $y=x-\frac{1}{3}x^3+\frac{1}{5}x^5-\frac{1}{7}x^7+\dots$
 9. $y=x-2x^2+3x^3-\dots$ 11. $y=x+\frac{x^2}{1 \times 2}+\frac{x^3}{1 \times 2 \times 3}+\dots$
 12. Find the fractions in the form $\frac{a+bx}{p+qx+rx^2}$ whose expansions produce the series:

$$1+3x+2x^2-x^3-\dots;$$

$$3+2x+3x^2+7x^3+\dots;$$

$$\frac{2}{3}-\frac{1}{18}x+\frac{1}{6}x^2-\frac{1}{54}x^3+\dots$$

Resolve into partial fractions:

13. $\frac{7x+1}{(x+4)(x-5)}$ 16. $\frac{x-2}{x^2-3x-10}$ 19. $\frac{3x^2-4}{x^2(x+5)}$
 14. $\frac{6}{(x+3)(x+4)}$ 17. $\frac{3}{x^3-1}$ 20. $\frac{7x^2-x}{(x-1)^2(x+2)}$
 15. $\frac{5x-1}{(2x-1)(x-5)}$ 18. $\frac{x^2-x-3}{x(x^2-4)}$ 21. $\frac{2x^2-7x+1}{x^3+1}$

CHAPTER XXVIII.

THE EXPONENTIAL THEOREM.

488. *To expand a^x in a series of ascending powers of x*

$$a^x = \{1 + (a-1)\}^x;$$

therefore, by the Binomial Theorem,

$$\begin{aligned} a^x = 1 + x(a-1) + \frac{x(x-1)}{1 \times 2} (a-1)^2 \\ + \frac{x(x-1)(x-2)}{1 \times 2 \times 3} (a-1)^3 + \dots \end{aligned}$$

By performing the indicated operations, this series will consist of powers of x , and the whole series can be rearranged in ascending powers of x .

The coefficient of x will be found to be

$$(a-1) - \frac{1}{2}(a-1)^2 + \frac{1}{6}(a-1)^3 - \frac{1}{24}(a-1)^4 + \dots,$$

which may be represented by A .

And if the coefficients of x^2, x^3, x^4, \dots be represented by B, C, D, \dots the series may be written

$$a^x = 1 + Ax + Bx^2 + Cx^3 + Dx^4 + \dots$$

Now, B, C, D, \dots may be found in terms of A ; for this series is true whatever be the value of x , since A, B, C, D, \dots are functions of a , and therefore wholly independent of the value of x . Hence,

$$a^{x+y} = 1 + A(x+y) + B(x+y)^2 + C(x+y)^3 + D(x+y)^4 + \dots$$

But $a^{x+y} = a^x \times a^y = a^y (1 + Ax + Bx^2 + Cx^3 + Dx^4 + \dots)$.

These two series are identically equal, and therefore, by § 487, the coefficient of x in the first series is equal to the coefficient of x in the second series; that is,

$$\begin{aligned}
 A + 2By + 3Cy^2 + 4Dy^3 + \dots &= Aa^y \\
 &= A(1 + Ay + By^2 + Cy^3 + Dy^4 + \dots) \\
 &= A + A^2y + AB y^2 + ACy^3 + \dots
 \end{aligned}$$

And since these expressions are identically equal for *all* values of y , the coefficients of y, y^2, y^3, \dots in the one are equal to the coefficients of y, y^2, y^3, \dots respectively, in the other.

$$\text{whence,} \quad 2B = A^2; \quad 3C = AB; \quad 4D = AC; \dots$$

$$\text{and} \quad B = \frac{A^2}{2}; \quad C = \frac{AB}{3}; \quad D = \frac{AC}{4}; \dots$$

$$\text{So that} \quad a^x = 1 + Ax + \frac{A^2x^2}{1 \times 2} + \frac{A^3x^3}{1 \times 2 \times 3} + \dots$$

$$\text{in which} \quad A = (a-1) - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \dots$$

489. If x is taken such that $Ax=1$; then $x=\frac{1}{A}$, and the last series becomes

$$a^{\frac{1}{A}} = 1 + 1 + \frac{1}{1 \times 2} + \frac{1}{1 \times 2 \times 3} + \dots$$

The value of this series can easily be computed to any degree of approximation, and is 2.7182818.....

This constant is denoted, for shortness, by the letter e , and is the number whose logarithm is 1 in the *Napierian* system of logarithms.

In this system no base is assumed, but a logarithm is defined to be such that the increment of the number shall be the product of the number by the increment of the logarithm.

490. Since $a^{\frac{1}{A}} = e$, $a = e^A$, and $A = \log_e a$; so that

$$a^x = 1 + (\log_e a)x + \frac{(\log_e a)^2 x^2}{1 \times 2} + \frac{(\log_e a)^3 x^3}{1 \times 2 \times 3} + \dots$$

491. The preceding proof has proceeded on the *assumption* that a is not greater than 2; for the sum of the series $(a-1) - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \dots$ is not convergent if a is greater than 2.

To extend the proof to cover all values of a , put b^y for a in the theorem

$$a^x = 1 + (\log_e a)x + \frac{(\log_e a)^2 x^2}{1 \times 2} + \frac{(\log_e a)^3 x^3}{1 \times 2 \times 3} + \dots$$

Now, by taking y small enough, b may be made as great as we please, while a is not greater than 2. Then

$$b^{yx} = 1 + (\log_e b)yx + \frac{(\log_e b)^2 y^2 x^2}{1 \times 2} + \frac{(\log_e b)^3 y^3 x^3}{1 \times 2 \times 3} + \dots$$

and by putting z for yx ,

$$b^z = 1 + (\log_e b)z + \frac{(\log_e b)^2 z^2}{1 \times 2} + \frac{(\log_e b)^3 z^3}{1 \times 2 \times 3} + \dots$$

Hence the series is true for all values of a .

492. In the theorem

$$a^x = 1 + (\log_e a)x + \frac{(\log_e a)^2 x^2}{1 \times 2} + \frac{(\log_e a)^3 x^3}{1 \times 2 \times 3} + \dots$$

put e for a , and observe that $\log_e e = 1$; § 294.

$$\text{then } e^x = 1 + x + \frac{x^2}{1 \times 2} + \frac{x^3}{1 \times 2 \times 3} + \dots,$$

a result true for *all* values of x .

COMPUTATION OF LOGARITHMS.

493. To expand $\log_e (1 + x)$ in a series of ascending powers of x .

Since, by § 491, $\log_e a = A = (a - 1) - \frac{1}{2}(a - 1)^2 + \frac{1}{3}(a - 1)^3 - \dots$, by putting $1 + x$ for a , and therefore x for $a - 1$,

$$\log_e (1 + x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots$$

494. This series is called the **Logarithmic Series**, but is not of a form for practical use, as may be seen by substituting 10 for x ; which gives

$$\log_e 11 = 10 - \frac{10^2}{2} + \frac{10^3}{3} - \frac{10^4}{4} + \dots$$

in which the values of the successive terms are increasing, and no number of them can be taken that will give an approximate value of $\log_e 11$.

It is necessary, therefore, to obtain from it other formulas, as follows :

$$\begin{aligned}
\log_e(1+x) &= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots \\
\log_e(1-x) &= -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 - \dots \text{ (by putting } -x \text{ for } x). \\
\therefore \log_e(1+x) - \log_e(1-x) &= 2x + 2\left(\frac{1}{3}x^3\right) + 2\left(\frac{1}{5}x^5\right) + \dots \\
&= 2\left(x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \dots\right).
\end{aligned}$$

$$\text{But } \log_e(1+x) - \log_e(1-x) = \log_e\left(\frac{1+x}{1-x}\right). \quad \S 308.$$

$$\therefore \log_e\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \dots\right).$$

In this series put $\frac{m-n}{m+n}$ for x , and therefore $\frac{m}{n}$ for $\frac{1+x}{1-x}$;

$$\text{then, } \log_e \frac{m}{n} = 2 \left\{ \frac{m-n}{m+n} + \frac{1}{3} \left(\frac{m-n}{m+n} \right)^3 + \frac{1}{5} \left(\frac{m-n}{m+n} \right)^5 + \dots \right\}.$$

In this last series put $n+1$ for m , and therefore $\frac{1}{2n+1}$ for $\frac{m-n}{m+n}$.

$$\begin{aligned}
\log_e \frac{n+1}{n} &= \log_e(n+1) - \log_e n \\
&= 2 \left\{ \frac{1}{2n+1} + \frac{1}{3} \left(\frac{1}{2n+1} \right)^3 + \dots \right\}.
\end{aligned}$$

$$\therefore \log_e(n+1) = \log_e n + 2 \left\{ \frac{1}{2n+1} + \frac{1}{3} \left(\frac{1}{2n+1} \right)^3 + \dots \right\};$$

by which the logarithm of any number can be obtained from that of the preceding number.

495. To compute logarithms to the base e for 1, 2, 3, 4, etc.,

$\log_e 1 = 0$. Hence, by the last series,

$$\begin{aligned}
\log_e(1+1) &= \log_e 2 = 0 + 2 \left\{ \frac{1}{3} + \frac{1}{3 \times 3^3} + \frac{1}{5 \times 3^5} + \frac{1}{7 \times 3^7} + \dots \right\} \\
&= 0.69314718\dots \text{ (by computation);}
\end{aligned}$$

$$\begin{aligned}
\log_e(2+1) &= \log_e 3 = \log_e 2 + 2 \left\{ \frac{1}{5} + \frac{1}{3 \times 5^3} + \frac{1}{5 \times 5^5} + \dots \right\} \\
&= 1.09861228\dots
\end{aligned}$$

$$\begin{aligned}
\log_e 4 &= 2 \log_e 2 \\
&= 1.38629436\dots \quad \S 305.
\end{aligned}$$

$$\begin{aligned}
\log_e 5 &= \log_e 4 + 2 \left\{ \frac{1}{9} + \frac{1}{3 \times 9^3} + \frac{1}{5 \times 9^5} + \dots \right\} \\
&= 1.60943791\dots
\end{aligned}$$

$$\begin{aligned}
\log_e 10 &= \log_e 5 + \log_e 2 \\
&= 2.30258509\dots
\end{aligned}$$

496. If N = any number,

$$\log_{10} N = \frac{\log_e N}{\log_e 10} = \log_e N \times \frac{1}{\log_e 10}. \quad \S \text{ 320.}$$

The multiplier $\frac{1}{\log_e 10}$, by means of which every natural logarithm may be changed to the corresponding decimal logarithm, is called the **Modulus** of the common system, and is represented by M . Hence,

$$M = \frac{1}{2.30258509.....} = 0.43429448.....$$

497. Common logarithms may be obtained from natural logarithms, as explained in § 496, or they may be computed directly by adapting to the common system the series

$$\log_e(n+1) = \log_e n + 2 \left\{ \frac{1}{2n+1} + \frac{1}{3} \left(\frac{1}{2n+1} \right)^3 + \dots \right\}.$$

Thus, $\log_{10}(n+1)$

$$= \log_{10} n + 2M \left\{ \frac{1}{2n+1} + \frac{1}{3} \left(\frac{1}{2n+1} \right)^3 + \frac{1}{5} \left(\frac{1}{2n+1} \right)^5 + \dots \right\};$$

and by means of this series a logarithm in the common system may be computed from that of the preceding number.

498. In finding the logarithm of a number which consists of more digits than are given in the tables, it is *assumed* that when the difference of two numbers is small in comparison with either of them, the difference of their logarithms is *proportional* to the difference of the numbers. The truth of this assumption may be shown as follows:

$$\begin{aligned} \log_{10}(n+d) - \log_{10} n &= \log_{10} \left(\frac{n+d}{n} \right) = \log_{10} \left(1 + \frac{d}{n} \right) \\ &= M \log_e \left(1 + \frac{d}{n} \right) = M \left(\frac{d}{n} - \frac{d^2}{2n^2} + \frac{d^3}{3n^3} - \dots \right). \end{aligned}$$

When d is very small in comparison with n , the terms of the right side, except the first, will be so small that they may be neglected;

so that
$$\log_{10}(n+d) - \log_{10} n = \frac{Md}{n},$$

and as M is constant and n a given number,

$$\{\log_{10}(n+d) - \log_{10} n\} \propto d.$$

CHAPTER XXIX.

THE DIFFERENTIAL METHOD.

499. IF we have a series whose terms proceed according to some law, but are not immediately equidifferent, we may find the difference of every two consecutive terms, and thus form a series of differences called the *first order of differences*. If the differences of the terms of this new series be similarly taken, another new series will be formed called the *second order of differences*, and so on.

Let a, b, c, d, \dots be the terms of the series.

Then $b - a, c - b, d - c, \dots$, which may be denoted by a_1, b_1, c_1, \dots , will be the first order of differences.

Again, $b_1 - a_1, c_1 - b_1, d_1 - c_1, \dots$, which may be denoted by a_2, b_2, c_2, \dots , will be the second order of differences.

This process may be continued as long as there are any differences.

The given series and the successive orders of differences arranged in lines will be:

$$\begin{array}{cccccccc}
 a & b & c & d & e & f & \dots \\
 a_1, & b_1 & c_1 & d_1 & e_1 & \dots \\
 a_2 & b_2 & c_2 & d_2 & \dots \\
 a_3 & b_3 & c_3 & \dots \\
 & \dots
 \end{array}$$

500. Let it be required to express the $(n + 1)th$ term of the series a, b, c, d, \dots in terms of a, a_1, a_2, \dots

As	$b - a = a_1, \therefore b = a + a_1;$	
as	$b_1 - a_1 = a_2, \therefore b_1 = a_1 + a_2;$	
as	$b_2 - a_2 = a_3, \therefore b_2 = a_2 + a_3;$	and so on.

In like manner, $c = b + b_1 = a + 2a_1 + a_2$,
 $c_1 = b_1 + b_2 = a_1 + 2a_2 + a_3$,
 $c_2 = b_2 + b_3 = a_2 + 2a_3 + a_4$, and so on.
 Likewise, $d = c + c_1 = a + 3a_1 + 3a_2 + a_3$, and so on.

The coefficients, therefore, of a, a_1, a_2, a_3, \dots , in the expressions for b, c, d, \dots , are the same as the coefficients obtained from the expansion of the expression $(a+b)^n$; and since by the Binomial Theorem the coefficients of $(a+b)^n$ are

$$1, n, \frac{n(n-1)}{1 \times 2}, \frac{n(n-1)(n-2)}{1 \times 2 \times 3}, \dots,$$

the $(n+1)$ th term of the series a, b, c, d, \dots will be

$$a + na_1 + \frac{n(n-1)}{1 \times 2} a_2 + \frac{n(n-1)(n-2)}{1 \times 2 \times 3} a_3 + \dots$$

501. Let it be required to find the sum of n terms of the series a, b, c, d, \dots in terms of a, a_1, a_2, \dots

The sum of two terms $= a + b$,
 of three terms $= a + b + c$,
 of four terms $= a + b + c + d$,

and so on; so that, if another series be formed,

$$0, a, a + b, a + b + c, \dots,$$

the $(n+1)$ th term of this series will be the sum of n terms of the series a, b, c, d, \dots

But, by the preceding proof, the $(n+1)$ th term of the series

$$0, a, a + b, a + b + c, \dots$$

is $0 + na + \frac{n(n-1)}{1 \times 2} a_1 + \frac{n(n-1)(n-2)}{1 \times 2 \times 3} a_2 + \dots$

Hence, the sum of n terms of the series a, b, c, d, \dots is

$$\begin{aligned} & na + \frac{n(n-1)}{1 \times 2} a_1 + \frac{n(n-1)(n-2)}{1 \times 2 \times 3} a_2 + \dots \\ &= n \left\{ a + \frac{n-1}{2} a_1 + \frac{(n-1)(n-2)}{2 \times 3} a_2 + \dots \right\}. \end{aligned}$$

- (1) Find the sum of the squares of the first n natural numbers, $1^2, 2^2, 3^2, 4^2, \dots, n^2$.

1	4	9	16	n^2	= given series.
3	5	7	9		= first order of differences.
2	2	2			= second order of differences.
0	0				= third order of differences.

Therefore, $a = 1$, $a_1 = 3$, $a_2 = 2$, $a_3 = 0$.

These values substituted in the general formula give

$$\begin{aligned}
 \text{Sum} &= n \left\{ 1 + \frac{n-1}{2} \times 3 + \frac{(n-1)(n-2)}{2 \times 3} \times 2 \right\} \\
 &= n \left\{ 1 + \frac{3n}{2} - \frac{3}{2} + \frac{1}{3}(n^2 - 3n + 2) \right\} \\
 &= \frac{n}{6} \{ 6 + 9n - 9 + 2n^2 - 6n + 4 \} \\
 &= \frac{n}{6} \{ 2n^2 + 3n + 1 \} = \frac{n(n+1)(2n+1)}{6}.
 \end{aligned}$$

If $n = 12$, the sum equals $2 \times 13 \times 25 = 650$.

- (2) Find the sum of the cubes of the first n natural numbers, $1^3, 2^3, 3^3, 4^3, 5^3, \dots, n^3$.

1	8	27	64	125	= given series.
7	19	37	61		= first order of differences.
12	18	24			= second order of differences.
6	6				= third order of differences.
0					= fourth order of differences.

Hence, $a = 1$, $a_1 = 7$, $a_2 = 12$, $a_3 = 6$, $a_4 = 0$.

These values substituted in the general formula give for the sum

$$\begin{aligned}
 n \left\{ 1 + \frac{(n-1)7}{2} + \frac{(n-1)(n-2)12}{2 \times 3} + \frac{(n-1)(n-2)(n-3)6}{2 \times 3 \times 4} \right\} \\
 &= \frac{n}{4} \{ 4 + 14n - 14 + 8n^2 - 24n + 16 + n^3 - 6n^2 + 11n - 6 \} \\
 &= \frac{n}{4} \{ n^3 + 2n^2 + n \} = \frac{n^2(n+1)^2}{4} = \left\{ \frac{n(n+1)}{2} \right\}^2.
 \end{aligned}$$

Now, $\frac{n(n+1)}{2}$ = the sum of the first n numbers, § 382; hence,

The sum of the cubes of the first n natural numbers is equal to the square of the sum of the numbers.

- (3) Find the twelfth term and the sum of 12 terms of the series 300, 270, 242, 216,

$$\begin{array}{rcccccl}
 300 & 270 & 242 & 216 & \dots & = \text{given series.} \\
 -30 & -28 & -26 & \dots & & = \text{first order of differences.} \\
 & 2 & 2 & \dots & & = \text{second order of differences.} \\
 & & 0 & \dots & & = \text{third order of differences.}
 \end{array}$$

Hence, $a = 300$, $a_1 = -30$, $a_2 = 2$, $a_3 = 0$, and $n = 12$.

These values substituted in the general formula give

$$\begin{aligned}
 \text{Twelfth term} &= 300 - 11 \times 30 + \frac{11 \times 10}{1 \times 2} \times 2 \\
 &= 300 - 330 + 110 \\
 &= 80.
 \end{aligned}$$

$$\begin{aligned}
 \text{Sum} &= 12 \left(300 - \frac{11}{2} \times 30 + \frac{11 \times 10}{2 \times 3} \times 2 \right) \\
 &= 12 (300 - 165 + 36\frac{2}{3}) \\
 &= 2080.
 \end{aligned}$$

EXERCISE 127.

- Find the fiftieth term of 1, 3, 8, 20, 43,
- Find the sum of the series 4, 12, 29, 55, to 20 terms.
- Find the twelfth term of 4, 11, 28, 55, 92,
- Find the sum of the series 43, 27, 14, 4, -3, to 12 terms.
- Find the seventh term of 1, 1.235, 1.471, 1.708,
- Find the sum of the series 70, 66, 62.3, 58.9, to 15 terms.
- Find the eleventh term of 343, 337, 326, 310,
- Find the sum of the series 7×13 , 6×11 , 5×9 , to 9 terms.
- Find the sum of n terms of the series 3×8 , 6×11 , 9×14 , 12×17 ,
- Find the sum of n terms of the series 1, 6, 15, 28, 45,

PILES OF SPHERICAL SHOT.

502. When the pile is in the form of a triangular pyramid, the summit consists of a single shot resting on three below; and these three rest on a course of six; and these six on a course of ten, and so on, so that the courses will form the series,

$$1, 1 + 2, 1 + 2 + 3, 1 + 2 + 3 + 4, \dots, 1 + 2 + \dots + n.$$

1	3	6	10	15	= given series.
2	3	4	5		= first order of differences.
1	1	1			= second order of differences.
0	0				= third order of differences.

Hence, $a = 1$, $a_1 = 2$, $a_2 = 1$, $a_3 = 0$.

These values substituted in the general formula give

$$\begin{aligned} \text{Sum} &= n \left\{ 1 + \frac{n-1}{2} \times 2 + \frac{(n-1)(n-2)}{2 \times 3} \right\} \\ &= n \left\{ 1 + n - 1 + \frac{n^2 - 3n + 2}{6} \right\} \\ &= \frac{n}{6} \{(n+1)(n+2)\} \\ &= \frac{n(n+1)(n+2)}{1 \times 2 \times 3}. \end{aligned}$$

in which n is the number of balls in the side of the bottom course.

503. When the pile is in the form of a pyramid with a square base, the summit consists of one shot, the next course consists of four balls, the next of nine, and so on. The number of shot, therefore, is the sum of the series,

$$1^2, 2^2, 3^2, 4^2, \dots, n^2.$$

Which, by § 501, is $\frac{n(n+1)(2n+1)}{1 \times 2 \times 3},$

in which n is the number of balls in the side of the bottom course.

504. When the pile has a base which is rectangular, but not square, the pile will terminate with a single row. Suppose p the number of shot in this row; then the second course will consist of $2(p+1)$ shot; the third course of $3(p+2)$; and the n th course of $n(p+n-1)$. Hence the series will be

$$p, 2p+2, 3p+6, \dots, n(p+n-1).$$

p	$2p+2$	$3p+6$	$4p+12$	\dots	= given series.
$p+2$	$p+4$	$p+6$	\dots		= first order of differences.
2	2	\dots			= second order of differences.
0	\dots				= third order of differences.

Hence, $a = p$, $a_1 = p+2$, $a_2 = 2$, $a_3 = 0$.

These values substituted in the general formula give

$$\begin{aligned}
 \text{Sum} &= n \left\{ p + \frac{n-1}{2} (p+2) + \frac{(n-1)(n-2)}{2 \times 3} \times 2 \right\} \\
 &= \frac{n}{6} \{ 6p + 3(n-1)(p+2) + 2(n-1)(n-2) \} \\
 &= \frac{n}{6} (6p + 3np - 3p + 6n - 6 + 2n^2 - 6n + 4) \\
 &= \frac{n}{6} (3np + 3p + 2n^2 - 2) \\
 &= \frac{n}{6} (n+1)(3p+2n-2).
 \end{aligned}$$

If n' denotes the number in the longest row, then $n' = p + n - 1$, and therefore $p = n' - n + 1$; and the formula may be written

$$\frac{n}{6} (n+1)(3n' - n + 1).$$

in which n denotes the number in the width, and n' in the length, of the bottom course.

505. When the pile is incomplete, compute the number in the pile as if complete, then the number in that part of the pile that is lacking, and take the difference of the results.

EXERCISE 128.

1. Determine the number of shot in the side of the base of a triangular pile which contains 286 shot.
2. The number of shot in the upper course of a square pile is 169, and in the lowest course 1089. How many shot are there in the pile ?
3. Find the number of shot in a rectangular pile having 17 shot in one side of the base and 42 in the other.
4. Find the number of shot in five courses of an incomplete triangular pile which has 15 in one side of the base.
5. The number of shot in a triangular pile is to the number in a square pile, of the same number of courses, as 22 : 41. Find the number of shot in each pile.
6. Find the number of shot required to complete a rectangular pile having 15 and 6 shot, respectively, in the sides of its upper course.
7. How many shot must there be in the lowest course of a triangular pile so that 10 courses of the pile, beginning at the base, may contain 37,020 shot ?
8. Find the number of shot in a complete rectangular pile of 15 courses which has 20 shot in the longest side of its base.
9. Find the number of shot in the bottom row of a square pile which contains 2600 more shot than a triangular pile of the same number of courses.
10. Find the number of shot in a complete square pile in which the number of shot in the base and the number in the fifth course above differ by 225.
11. Find the number of shot in a rectangular pile which has 600 in the lowest course and 11 in the top row.

SERIES OF SEPARABLE TERMS.

506. It is evident from the appearance of certain series that they are the sums or the differences of two other series.

(1) Find the sum of the series

$$\frac{1}{1 \times 2}, \frac{1}{2 \times 3}, \frac{1}{3 \times 4}, \dots, \frac{1}{n(n+1)}.$$

Each term of this series may evidently be expressed in two parts:

$$\frac{1}{1} - \frac{1}{2}, \frac{1}{2} - \frac{1}{3}, \dots, \frac{1}{n} - \frac{1}{n+1};$$

so that the sum will be

$$\left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \dots + \left(\frac{1}{n} - \frac{1}{n+1}\right),$$

in which the second part of each term, except the last, is cancelled by the first part of the next succeeding term.

$$\text{Hence, the sum equals } 1 - \frac{1}{n+1}.$$

If n increases without limit, $\frac{1}{n+1}$ approaches 0 as a limit, and the sum equals 1.

(2) Find the sum of the series

$$\frac{1}{3 \times 5}, \frac{1}{4 \times 6}, \frac{1}{5 \times 7}, \dots, \frac{1}{n(n+2)}.$$

Each term may be written,

$$\frac{1}{2} \left(\frac{1}{3} - \frac{1}{5} \right), \frac{1}{2} \left(\frac{1}{4} - \frac{1}{6} \right), \dots, \frac{1}{2} \left(\frac{1}{n} - \frac{1}{n+2} \right).$$

\therefore Sum

$$\begin{aligned} &= \frac{1}{2} \left(\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \dots + \frac{1}{n} - \frac{1}{5} - \frac{1}{6} - \dots - \frac{1}{n} - \frac{1}{n+1} - \frac{1}{n+2} \right) \\ &= \frac{1}{2} \left(\frac{1}{3} + \frac{1}{4} - \frac{1}{n+1} - \frac{1}{n+2} \right). \end{aligned}$$

$$\text{Hence, the sum equals } \frac{7}{24} - \frac{1}{2(n+1)} - \frac{1}{2(n+2)}.$$

If n increases without limit, the sum equals $\frac{7}{24}$.

(3) Find the sum of the series

$$\frac{1}{3 \times 8}, \frac{1}{6 \times 12}, \frac{1}{9 \times 16}, \dots, \frac{1}{3n(4n+4)}.$$

By multiplying each term by 12 the series becomes

$$\frac{1}{1 \times 2}, \frac{1}{2 \times 3}, \frac{1}{3 \times 4}, \dots, \frac{1}{n(n+1)}.$$

which is the same as the series in Ex. (1).

Hence, when n increases without limit, the sum equals $\frac{1}{12}$.

(4) Find the sum of the series

$$\frac{4}{1 \times 2 \times 3}, \frac{5}{2 \times 3 \times 4}, \frac{6}{3 \times 4 \times 5}, \dots, \frac{n+3}{n(n+1)(n+2)}.$$

To determine whether the terms of this series can each be separated into two parts, assume

$$\frac{n+3}{n(n+1)(n+2)} = \frac{A}{n(n+1)} + \frac{B}{(n+1)(n+2)}.$$

Reduce the right member to a common denominator; then

$$\frac{n+3}{n(n+1)(n+2)} = \frac{(A+B)n+2A}{n(n+1)(n+2)}.$$

$$\therefore 2A = 3 \text{ and } A + B = 1.$$

§ 487.

$$\text{Whence } A = \frac{3}{2} \text{ and } B = -\frac{1}{2}.$$

Hence, each term may be separated into two terms, and the series will then become

$$\begin{aligned} & \frac{3}{2} \left(\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \dots + \frac{1}{n(n+1)} \right) \\ & - \frac{1}{2} \left(\frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \dots + \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} \right) \\ & = \frac{3}{4} + \left(\frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \dots + \frac{1}{n(n+1)} \right) - \frac{1}{2(n+1)(n+2)} \\ & = \frac{3}{4} + \left\{ \left(\frac{1}{2} - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{4} \right) + \dots + \left(\frac{1}{n} - \frac{1}{n+1} \right) \right\} - \frac{1}{2(n+1)(n+2)} \\ & = \frac{3}{4} + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{2(n+1)(n+2)} \\ & = \frac{5}{4} - \frac{3n+5}{2(n+1)(n+2)}. \end{aligned}$$

If n increases without limit, the sum equals $\frac{5}{4}$.

EXERCISE 129.

Sum to n terms, and to infinity, the following series :

1. $\frac{1}{1 \times 4}, \frac{1}{2 \times 5}, \frac{1}{3 \times 6}, \dots$
2. $\frac{1}{1 \times 3 \times 5}, \frac{1}{2 \times 4 \times 6}, \frac{1}{3 \times 5 \times 7}, \dots$
3. $\frac{1}{2 \times 4 \times 6}, \frac{1}{4 \times 6 \times 8}, \frac{1}{6 \times 8 \times 10}, \dots$
4. $\frac{4}{2 \times 3 \times 4}, \frac{7}{3 \times 4 \times 5}, \frac{10}{4 \times 5 \times 6}, \dots$
5. $\frac{1}{1 \times 2 \times 3}, \frac{1}{2 \times 3 \times 4}, \frac{1}{3 \times 4 \times 5}, \dots$

INTERPOLATION OF SERIES.

507. As the expansion of $(a+b)^n$ by the Binomial Theorem has the same form for fractional as for integral values of n , so the formula

$$a + na_1 + \frac{n(n-1)}{1 \times 2} a_2 + \dots$$

may be extended to cases in which n is a fraction, and be employed to insert or *interpolate* terms in a series at required intervals between the given terms.

- (1) The cube roots of 27, 28, 29, 30, are 3, 3.03659, 3.07232, 3.10723. Find the cube root of 27.9.

$$\begin{array}{lll} 0.03659 & 0.03573 & 0.03491 = \text{first order of differences.} \\ -0.00086 & -0.00082 & = \text{second order of differences.} \\ 0.00004 & & = \text{third order of differences.} \end{array}$$

These values substituted in the general formula give

$$\begin{aligned} 3 + \frac{9}{10}(0.03659) - \frac{9}{10}\left(-\frac{1}{10}\right)\left(\frac{0.00086}{2}\right) + \frac{9}{10}\left(-\frac{1}{10}\right)\left(-\frac{11}{10}\right)\left(\frac{0.00004}{6}\right) \\ = 3 + 0.032931 + 0.0000387 + 0.00000066 \\ = 3.03297. \end{aligned}$$

- (2) Given $\log 127 = 2.1038$, $\log 128 = 2.1072$, $\log 129 = 2.1106$. Find $\log 127.37$.

0.0034 0.0034 = first order of differences.

0 = second order of differences.

Therefore, the differences of the second order will vanish, and the required logarithm will be

$$\begin{aligned} & 2.1038 + \frac{37}{100} \text{ of } 0.0034 \\ & = 2.1038 + 0.001358 \\ & = 2.1052. \end{aligned}$$

- (3) The latitude of the moon on a certain Monday at noon was $1^\circ 53' 18.9''$, at midnight $2^\circ 27' 8.6''$; on Tuesday at noon $2^\circ 58' 55.2''$, at midnight $3^\circ 28' 5.8''$; on Wednesday at noon $3^\circ 54' 8.8''$. Find its latitude at 9 P.M. on Monday.

The series expressed in seconds, and the differences, will be

$$\begin{array}{cccccc} 6798.9 & 8828.6 & 10735.2 & 12485.8 & 14048.8 & \\ 2029.7 & 1906.6 & 1750.6 & 1563 & & \\ -123.1 & -156 & -187.6 & & & \\ -32.9 & -31.6 & & & & \\ & 1.3 & & & & \end{array}$$

As 9 hours = $\frac{3}{4}$ of 12 hours, $n = \frac{3}{4}$.

Also, $a = 6798.9$, $a_1 = 2029.7$, $a_2 = -123.1$, $a_3 = -32.9$, $a_4 = 1.3$.

These values substituted in the general formula

$$\begin{aligned} & a + na_1 + \frac{n(n-1)}{1 \times 2} a_2 + \frac{n(n-1)(n-2)}{1 \times 2 \times 3} a_3 \\ & \quad + \frac{n(n-1)(n-2)(n-3)}{1 \times 2 \times 3 \times 4} a_4 + \dots \\ \text{give } & 6798.9 + \frac{3}{4}(2029.7) - \frac{3}{4}\left(-\frac{1}{4}\right)\left(\frac{123.1}{2}\right) \\ & \quad - \frac{3}{4}\left(-\frac{1}{4}\right)\left(-\frac{5}{4}\right)\left(\frac{32.9}{6}\right) + \dots \\ & = 6798.9 + 1522.27 + 11.54 - 1.29 \\ & = 8331.4 = 2^\circ 18' 51.4''. \end{aligned}$$

CHAPTER XXX.

THE THEORY OF NUMBERS.

SYSTEMS OF NOTATION.

508. A **System of Notation** is a method of expressing numbers by means of a series of powers of some fixed number, called the **Radix**, or **Base**, of the scale in which the different numbers are expressed.

509. Integral numbers expressed in any system are polynomials arranged according to the descending powers of the base, and containing multiples of these powers written in a condensed form by omitting the exponents and indicating them by the *place* of the digits.

Thus, 2384 in the decimal system means

$$2 \times 10^3 + 3 \times 10^2 + 8 \times 10 + 4;$$

and in the nonary system means

$$2 \times 9^3 + 3 \times 9^2 + 8 \times 9 + 4.$$

510. If r be any integer, any integral number N may be expressed in the form

$$N = ar^n + br^{n-1} + \dots + pr^2 + qr + s,$$

in which the coefficients a, b, c, \dots , are each less than r .

For, divide N by r^n , the highest power of r contained in N , and let the quotient be a with the remainder N_1 .

Then, $N = ar^n + N_1$.

In like manner, $N_1 = br^{n-1} + N_2$, $N_2 = cr^{n-2} + N_3$; and so on.

By continuing this process, a remainder s will at length be reached which is less than r . So that,

$$N = ar^n + br^{n-1} + \dots + pr^2 + qr + s.$$

Some of the coefficients s, q, p, \dots may vanish, and each will be less than r ; that is, their values may range from zero to $r-1$. Hence, including zero, the scale of r will contain r digits.

511. *To express any integral number N in the scale of r .*

It is required to express N in the form of

$$ar^n + br^{n-1} + \dots + pr^2 + qr + s,$$

and to show how the digits a, b, \dots may be found.

If
$$N = ar^n + br^{n-1} + \dots + pr^2 + qr + s,$$

then
$$\frac{N}{r} = ar^{n-1} + br^{n-2} + \dots + pr + q + \frac{s}{r}.$$

That is, the remainder on dividing N by r is s , the *last* digit.

Let
$$N_1 = ar^{n-1} + br^{n-2} + \dots + pr + q,$$

then
$$\frac{N_1}{r} = ar^{n-2} + br^{n-3} + \dots + p + \frac{q}{r}.$$

That is, the remainder is q , the *last but one* of the digits.

Hence, to express an integral number in a proposed scale,

Divide the number by the radix, then the quotient by the radix, and so on; the successive remainders will be the successive digits beginning from the units' place.

(1) Express 42897 (common scale) in the senary scale (scale of six), and transform 37214 from the octonary scale (scale of eight) to the nonary scale (scale of nine).

$$\begin{array}{r} \text{(i.) } 6 \overline{)42897} \\ \underline{6 \overline{)7149} \dots 3} \\ 6 \overline{)1191} \dots 3 \\ \underline{6 \overline{)198} \dots 3} \\ 6 \overline{)33} \dots 0 \\ 5 \dots 3 \end{array}$$

Ans. 530333.

$$\begin{array}{r} \text{(ii.) } 9 \overline{)37214} \\ \underline{9 \overline{)3363} \dots 1} \\ 9 \overline{)305} \dots 6 \\ \underline{9 \overline{)25} \dots 8} \\ 2 \dots 3 \end{array}$$

Ans. 23861.

In (ii.) the radix is 8; and hence the two digits on the left, 37, do not mean *thirty-seven*, but $3 \times 8 + 7$, or *thirty-one*, which contains 9 three times, with the remainder 4.

The next partial dividend is $4 \times 8 + 2 = 34$, which contains 9 three times, with the remainder 7; and so on.

- (2) What is the radix of the scale in which the number 140 (common scale) is expressed by 352?

Let r denote the required radix ;
 then the value of $352 = 3r^2 + 5r + 2$.
 Hence, $3r^2 + 5r + 2 = 140$.
 Whence, $r = 6$. *Ans.*

512. As in the decimal scale it is convenient to express quantities less than the unit by means of decimal fractions, so *radix fractions* may be employed in any other scale.

Thus, 0.2341 in the decimal scale means

$$\frac{2}{10} + \frac{3}{10^2} + \frac{4}{10^3} + \frac{1}{10^4}.$$

And in a scale whose radix is r it means

$$\frac{2}{r} + \frac{3}{r^2} + \frac{4}{r^3} + \frac{1}{r^4}.$$

513. Computations are made with numbers in any scale, by observing that *one unit of any order is equal to the radix-number of units of the next lower order; and that the radix-number of units of any order is equal to one unit of the next higher order.*

514. The practice of many centuries has decided in favor of the decimal system. For, when the radix is a small number, as 2 or 3, large numbers are expressed in a form so extended that they are not easily comprehended; and when the radix is a large number, the series of separate symbols that must be used is inconveniently large.

515. In some cases different systems are employed for expressing the same kind of quantity.

Thus, in measuring time, the smallest unit is the *second*; the next higher unit is the *minute*, which is equal to 60 seconds; the next higher unit is the *hour*, which is equal to 60 minutes; while the next higher unit is the *day*, which is equal to 24 hours.

EXERCISE 130.

1. If 6, 7, 8, 3, 2 are the digits of a number in the scale of r , beginning from the right, write the algebraical value of the number.
2. Find the product of 234 and 125 when r is the base of the scale.
3. In what scale will the common number 756 be expressed by 530 ?
4. In what scale will 540 be the square of 23 ?
5. Show that 1234321 will in any scale be a perfect square, and find its square root.
6. In what scale will 212, 1101, 1220 be in arithmetical progression ?
7. Multiply 31.24 by 0.31 in the scale of 5.
8. Find the least multiplier of 13168 which will make the product a perfect cube.

THE COMMON SYSTEM OF NOTATION.

516. A number which contains the factor a , since it is a multiple of a , may be written ma .

517. An even number, since it contains the factor 2, may be written $2m$; and an odd number may be written $2m + 1$.

518. A number which ends in zero, since it is a multiple of 10, may be written $10m$. A number which is a multiple of 100 may be written $100m$; and so on.

519. *If a prime number p is a factor of a product ab , and is not a factor a , it is a factor of b .*

$$\begin{array}{r} p) a (q_1 \\ \underline{pq_1} \\ r_1) p (q_2 \\ \underline{r_1q_2} \\ r_2 \text{ etc.} \end{array}$$

In finding the G.C.M. of p and a , let q_1, q_2, \dots denote the successive quotients, and r_1, r_2, \dots the successive remainders.

Then $a = pq_1 + r_1$; $p = q_2r_1 + r_2$; etc.

By multiplying each of these equations by b ,

$$ab = bpq_1 + br_1; bp = bq_2r_1 + br_2; \text{ etc.}$$

Hence, $br_1 = ab - bpq_1$;

and since p is a factor of ab and of bpq_1 , it is a factor of their difference br_1 .

Also, since p is a factor of br_1 , it is a factor of bq_2r_1 .

Again, $br_2 = bp - bq_2r_1$; and since p is a factor of bp and of bq_2r_1 , it is a factor of their difference br_2 .

In like manner it may be shown that p is a factor of br_3, br_4, br_5 , and so on.

Now, since p is a *prime number*, it has no common factor with a , except unity; and the last remainder, in finding the G.C.M. of p and a , will be 1.

Hence, p will be a factor of $b \times 1$; that is, a factor of b .

520. From the above proposition it follows that,

A prime number which is not a factor of either of two other numbers is not a factor of their product.

In other words,

The product of two or more prime numbers will contain no prime factor except themselves.

Hence,

A composite number can be separated into only one set of prime factors.

If two numbers are prime to one another, every power of one will be prime to every power of the other.

521. A fraction $\frac{a}{b}$ in its lowest terms cannot be equal to another fraction $\frac{c}{d}$ unless d is a multiple of b .

For, if $\frac{a}{b} = \frac{c}{d}$, then, by multiplying by d , $\frac{ad}{b} = c$; and since c is integral, $\frac{ad}{b}$ is integral.

But b is prime to a , and must therefore be contained in d an integral number of times.

522. From the last proposition it follows that,

A common fraction in its lowest terms will not produce a terminating decimal if its denominator contains any prime factor except 2 and 5.

For a terminating decimal is equivalent to a fraction with a denominator 10^n . Therefore, a fraction $\frac{a}{b}$ in its lowest terms cannot be equal to such a fraction, unless 10^n is a multiple of b . But 10^n , that is, $2^n \times 5^n$, contains no factors besides 2 and 5, and hence cannot be a multiple of b , if b contains any factors except these.

523. *If a square number is resolved into its prime factors, the exponent of each factor will be even.*

For, if any number $N = a^p \times b^q \times c^r \dots$,
 $N^2 = a^{2p} \times b^{2q} \times c^{2r} \dots$

524. Conversely: A number which has the exponents of all its prime factors even will be a perfect square; therefore, to change any number to a perfect square,

Resolve the number into its prime factors, select the factors which have odd exponents, and multiply the given number by the product of these factors.

Thus, to find the least number by which 250 must be multiplied to make it a perfect square.

$250 = 2 \times 5^3$, in which 2 and 5 are the factors which have odd exponents.

Hence, the number required is $2 \times 5 = 10$.

DIVISIBILITY OF NUMBERS.

525. *If two numbers N and N_1 , when divided by the same number a , have the same remainder r , their difference is divisible by a .*

For, if N when divided by a has a quotient q and a remainder r , then

$$N = qa + r.$$

And if N_1 when divided by a has a quotient q_1 and a remainder r , then

$$N_1 = q_1a + r.$$

Therefore, $N - N_1 = (q - q_1)a$.

526. *If the difference of two numbers N and N_1 is divisible by a , then N and N_1 when divided by a will have the same remainder.*

For, if $N - N_1 = (q - q_1)a$,

then $\frac{N}{a} - \frac{N_1}{a} = q - q_1$.

Therefore, $\frac{N}{a} - q = \frac{N_1}{a} - q_1$.

That is, $N - aq = N_1 - aq_1$.

527. *If two numbers N and N_1 , when divided by a given number a , have remainders r and r_1 , then NN_1 and rr_1 when divided by a will have the same remainder.*

For, if $N = qa + r$,
and $N_1 = q_1a + r_1$,
then $NN_1 = qq_1a^2 + qar_1 + q_1ar + rr_1$
 $= (qq_1a + qr_1 + q_1r)a + rr_1$.

Therefore, § 526, NN_1 and rr_1 when divided by a will have the same remainder.

As a particular case, 37 and 47 when divided by 7 have remainders 2 and 5, respectively.

Now $37 \times 47 = 1739$, and $2 \times 5 = 10$.

The remainder, when each of these two numbers is divided by 7, will be 3.

528. From the preceding propositions it follows that:

A number is divisible by 2, 4, 8, if the numbers denoted by its last digit, last two digits, last three digits, are divisible respectively by 2, 4, 8,

A number is divisible by 5, 25, 125, if the numbers denoted by its last digit, last two digits, last three digits, are divisible respectively by 5, 25, 125,

If from a number the sum of its digits is subtracted, the remainder will be divisible by 9.

For, if from a number expressed in the form of

$$a + 10b + 10^2c + 10^3d + \dots$$

$$a + b + c + d + \dots \quad \text{is subtracted,}$$

the remainder will be $(10 - 1)b + (10^2 - 1)c + (10^3 - 1)d + \dots$,
and $10 - 1, 10^2 - 1, 10^3 - 1, \dots$ will be a series of 9's.

Therefore, the remainder is divisible by 9.

Hence a number N may be expressed in the form of

$$9n + s \text{ (if } s \text{ denotes the sum of its digits);}$$

and N will be divisible by 3 if s is divisible by 3; and also by 9 if s is divisible by 9.

A number will be divisible by 11 if the difference between the sum of its digits in the even places and the sum of its digits in the odd places is 0 or a multiple of 11.

For, a number N expressed by digits (beginning from the right) a, b, c, d, \dots may be put in the form of

$$N = a + 10b + 10^2c + 10^3d + \dots$$

$$\therefore N - a + b - c + d - \dots = (10 + 1)b + (10^2 - 1)c + (10^3 + 1)d + \dots$$

But $10 + 1$ is a factor of $10 + 1, 10^2 - 1, 10^3 + 1, \dots$

Therefore, $N - a + b - c + d - \dots$ is divisible by $10 + 1 = 11$.

Hence, the number N may be expressed in the form of

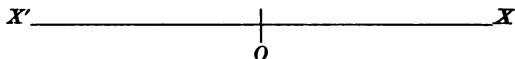
$$11n + (a + c + \dots) - (b + d + \dots),$$

and will be a multiple of 11 if $(a + c + \dots) - (b + d + \dots)$ is 0 or a multiple of 11.

CHAPTER XXXI.

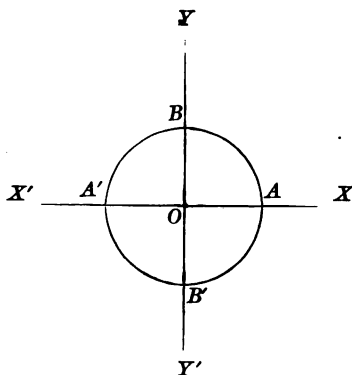
IMAGINARY NUMBERS.

529. ANY real number, whether positive or negative, integral or fractional, rational or surd, may be represented in magnitude by a horizontal line extending from the zero-point to the place it occupies.



Thus, if the line $X'X$ represent a horizontal line of unlimited length, and O be the zero-point, OX the positive, and OX' the negative direction, then all positive real numbers will have their place in OX , and all negative real numbers in OX' .

530. If the straight line OA , which may be supposed



to represent b units of length, turns around O 180° , it comes into the negative direction OA' ; $+b$ becomes $-b$. After a further turn of 180° , it comes again in the positive direction OA , and $-b$ becomes $+b$.

The turn of 180° signifies, then, the multiplication by the factor -1 . The double turn of 180°

signifies the multiplication twice by the factor -1 .

Since -1 is the product of $\sqrt{-1} \times \sqrt{-1}$, the turn of one-half of 180° signifies the multiplication by the factor $\sqrt{-1}$.

Hence, a definite geometrical meaning may be assigned to such an expression as $b\sqrt{-1}$.

Draw YY' through O , perpendicular to $X'X$; then $b\sqrt{-1}$ may be represented by b units counted on a line perpendicular to the original straight line at the zero-point. And the point B will be the place which this imaginary number occupies on the perpendicular.

Multiplying again by the factor $\sqrt{-1}$ brings the line OB to the position OA' , opposite OA , and changes $+b\sqrt{-1}$ to $-b$. Multiplying a third time by the factor $\sqrt{-1}$ brings the line into the position OB' and changes $-b$ to $-b\sqrt{-1}$. Multiplying by the factor $\sqrt{-1}$ a fourth time brings the line into its original position OA , and changes $-b\sqrt{-1}$ to $+b$, which corresponds to the laws governing the powers of $\sqrt{-1}$. § 281.

Hence, the *direction factors* (as they may be called), $+1$, $+\sqrt{-1}$, -1 , $-\sqrt{-1}$, indicate that b units are to be counted from O in the directions OX , OY , OX' , OY' , respectively. To the positive real number $+b$ the negative real number $-b$ is opposed; to the positive imaginary number $+b\sqrt{-1}$ the negative imaginary number $-b\sqrt{-1}$ is opposed.

531. The expression $\sqrt{-1}$ is called the **Imaginary Unit**, and every pure imaginary number can be expressed in the form of $\pm b\sqrt{-1}$.

For convenience the letter i is used for $\sqrt{-1}$. Hence,

$$\begin{aligned} i &= \sqrt{-1}, \\ i^2 &= (\sqrt{-1})^2 = -1, \\ i^3 &= -1 \sqrt{-1} = -\sqrt{-1}, \\ i^4 &= i^2 \times i^2 = -1 \times -1 = +1. \end{aligned}$$

OPERATIONS WITH IMAGINARY NUMBERS.

532. In order to add two imaginary numbers, we begin at the place which the first number occupies and count in the direction of the second number as many units as are equal to the absolute value of the second number. Therefore,

$$ai + bi = (a + b)i.$$

Hence, the sum of two imaginary numbers is also imaginary.

It follows that

$$ai + bi = bi + ai.$$

533. In order to find the difference between two imaginary numbers, we begin at the place that the minuend occupies, and count in the direction opposite to that of the subtrahend as many units as are equal to the absolute value of the subtrahend. Therefore,

$$ai - bi = (a - b)i.$$

Hence, the difference of two unequal imaginary numbers is also imaginary.

534. To multiply an imaginary number by a real number means to form from the multiplicand a new number, in the same way that the multiplier is formed from the real unit.

Therefore, $ai \times b = abi$.

For b is formed from the positive real unit by taking this unit b times to be added; ai must, therefore, be taken b times to be added.

That is, $ai \times b = ai + ai + ai + \dots$ taken b times $= abi$.

Again, $a \times bi = abi$.

For bi is formed from the positive real unit by forming the imaginary unit i from the positive real unit, and taking this imaginary unit b times to be added.

Therefore, from the multiplicand a the corresponding imaginary number ai must be formed, and ai taken b times to be added.

Hence, $a \times bi = ai \times b = abi$.

Also, $ai \times bi = -ab$.

For $ai \times bi = ai^2 \times b = -a \times b = -ab$.

Hence, the product of an imaginary number and a real number is imaginary; the product of two imaginary numbers is real.

It follows, also, that

$$ai \times b = b \times ai,$$

and $ai \times bi = bi \times ai$.

535. If $ab = d$ is substituted in these three equations,

then $a = \frac{d}{b}$, and $\frac{d}{b}i \times b = di$;

$$\frac{d}{b} \times bi = di, \quad \frac{d}{b}i \times bi = -d, \quad \text{or} \quad -\frac{d}{b}i \times bi = d.$$

Whence it follows from the general notion of division,

$$\frac{di}{b} = \frac{d}{b}i, \quad \frac{di}{bi} = \frac{d}{b}, \quad \frac{d}{bi} = -\frac{d}{b}i.$$

Hence, when the dividend and divisor are the one real and the other imaginary, the quotient is imaginary; when both are imaginary, the quotient is real.

From the second of these equations it is evident that a fraction remains unchanged if both numerator and denominator be multiplied by i .

536. Since $i^2 = -1$, $i^3 = -i$, $i^4 = +1$, it follows, in general, when n signifies any positive whole number,

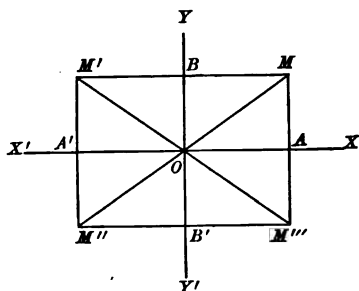
$$i^{4n} = +1, \quad i^{4n+1} = +i, \quad i^{4n+2} = -1, \quad i^{4n+3} = -i;$$

that is, $(ai)^n = ai \times ai \times ai \dots n \text{ times} = a^n i^n$.

So that the power of an imaginary number is real or imaginary according as the corresponding power of i is real or imaginary.

537. The sum of a real and an imaginary number, as $a + bi$, is called a **Complex Number**, in distinction from a *pure imaginary number* bi . Two complex numbers of the form $a + bi$ and $a - bi$ are called **Conjugate**.

538. In order to represent the complex number $a + bi$,



let $X'X$ be the line of real numbers, O the zero-point, and YY' perpendicular to $X'X$ at O be the line of imaginary numbers. Let A be the place of the real number a , B the place of the imaginary number bi .

Through A draw a line parallel to YY' , and through B draw a line parallel to $X'X$. Then M , the point of intersection, shows the place of the complex number $a + bi$; and the straight line OM represents the absolute value of this complex number.

The point M can also be reached by laying off a units from O on the line OX , and erecting at A , the last division point on this line, a perpendicular, and laying off b units on this perpendicular.

In like manner, the complex numbers $-a + bi$, $-a - bi$, $a - bi$, correspond respectively to the points M' , M'' , M''' .

539. If a and b have all values from $-\infty$ to $+\infty$ the point M will have every position in the infinite plane determined by $X'X$ and YY' .

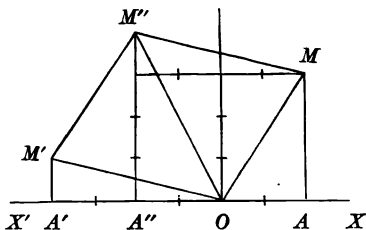
The expression $a + bi$ is then the general expression for all numbers. This expression includes zero when $a = 0$ and $b = 0$; includes all real numbers when $b = 0$; all imaginary numbers when $a = 0$; all complex numbers when a and b both differ from 0.

OPERATIONS WITH COMPLEX NUMBERS.

540. In order to add two complex numbers, $a + bi$ and $c + di$, it is necessary to proceed from the place occupied by the first number in the direction of the second number $c + di$, as far as the absolute value of the second number. The sum thus reached consists of the sum of the real and the sum of the imaginary parts of the two numbers.

Find the sum of $(2 + 3i) + (-4 + i)$.

$2 + 3i = OM$ and $-4 + i = OM'$ (§ 538). If now we proceed from M , the extremity of OM , in the direction of OM' as far as the absolute value of OM' , we reach the point M'' . Hence, $OM'' = -2 + 4i$, the sum of the two given complex numbers.



The same result is reached if we first find the value of $2 + (-4) = -2$. That is, if we count from O two real units to A'' , and add to this sum $3i + i = 4i$; that is, count four imaginary units from A'' on the perpendicular $A''M''$.

In general, the sum of two complex numbers is a complex number.

The sum of two conjugate numbers is a real number.

$$(a + bi) + (a - bi) = 2a.$$

541. To subtract $c + di$ from $a + bi$ it is necessary to proceed from the place occupied by the minuend in a direction opposite to that of the subtrahend as far as the absolute value of the subtrahend. The value thus reached is the difference of the real and the difference of the imaginary numbers.

Thus, $(a + bi) - (c + di) = (a - c) + (b - d)i$.

In general, the difference of two complex numbers is a complex number.

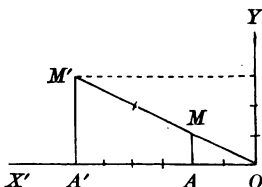
542. If $a + bi = c + di$, then must $a = c$ and $b = d$; for otherwise $a - c = (d - b)i$, or $\frac{a - c}{d - b} = i$; that is, a real number equal to an imaginary number, which is impossible.

543. If $a + bi = 0$, then must $a = 0$, and $b = 0$; for, otherwise $-\frac{a}{b} = i$, which is impossible.

544. In order to multiply a complex number $a + bi$ by a real number c , the number $a + bi$ must be taken c times, so that the real and the imaginary parts must be multiplied by c . That is,

$$(a + bi) \times c = (a + bi) + (a + bi) + \cdots c \text{ times} = ac + bci.$$

Thus, to multiply $-2 + i$ by 3: Take $OA = -2$ on OX' , and erect at A the perpendicular $AM = 1$. Then $OM = -2 + i$; and, by taking OM three times, the result is $OM' = -6 + -3i$ the product of $(-2 + i)$ by 3.



545. In order to multiply $a + bi$ by $c + di$ it is necessary to obtain a number from $a + bi$ in the same way that $c + di$ is obtained from the positive real unit. It is necessary, therefore, to take $a + bi$ itself c times, then to take $a + bi$ in the imaginary

$$\text{Again, } \frac{c(a+bi)}{d(a+bi)} = \frac{c}{d}.$$

That is, a quotient is unchanged if both the dividend and divisor are multiplied by the same complex number.

547. In order to divide one complex number by another, it is only necessary to multiply the dividend and divisor by the conjugate of the divisor, and the result is a complex number to be divided by a real number. Thus,

$$\begin{aligned} \frac{a+bi}{c+di} &= \frac{(a+bi)(c-di)}{(c+di)(c-di)} = \frac{(ac+bd) + (bc-ad)i}{c^2+d^2} \\ &= \frac{ac+bd}{c^2+d^2} + \frac{(bc-ad)i}{c^2+d^2}. \end{aligned}$$

The quotient of two complex numbers is a complex number. Thus,

$$\frac{3+i}{2+5i} = \frac{(3+i)(2-5i)}{(2+5i)(2-5i)} = \frac{11-13i}{29}.$$

548. The power of a complex number is a complex number. Thus,

$$\begin{aligned} (a+bi)^2 &= (a+bi)(a+bi) = (a^2-b^2) + 2abi, \\ (a+bi)^3 &= (a+bi)^2(a+bi) \\ &= (a^3-3ab^2) + (3a^2b-b^3)i, \quad \text{and so on.} \end{aligned}$$

549. The meaning of an *imaginary exponent* is determined by subjecting it to the same operations as if it were a real exponent.

It follows that such an expression as k^{a+bi} , where k is a real number and $a+bi$ a complex exponent, may be simplified by resolving it into two factors, one of which is a real number, and the other an imaginary power of e (e being used as in § 489).

By the ordinary rules for exponents,

$$k^{a+bi} = k^a \times k^{bi} = k^a [k^b]^i.$$

Put $k^b = e^u$, u denoting $\log[k^b]$ in the natural system; then

$$k^{a+bi} = k^a \times e^{ui}.$$

The value of e^{ui} is (§ 492)

$$e^{ui} = 1 + ui + \frac{u^2 i^2}{2} + \frac{u^3 i^3}{3} + \frac{u^4 i^4}{4} + \frac{u^5 i^5}{5} + \dots$$

This value (§§ 531 and 536) may be resolved into two series, as follows:

$$e^{ui} = 1 - \frac{u^2}{2} + \frac{u^4}{4} - \frac{u^6}{6} + \dots + i \left(u - \frac{u^3}{3} + \frac{u^5}{5} - \dots \right).$$

The series

$$1 - \frac{u^2}{2} + \frac{u^4}{4} - \frac{u^6}{6} + \dots$$

is a function of u called **Cosine** of u , and is written $\cos u$.

The series

$$u - \frac{u^3}{3} + \frac{u^5}{5} - \dots$$

is a function of u called **Sine** of u , and is written $\sin u$.

550. Hence, the value of e^{ui} may be written

$$e^{ui} = \cos u + i \sin u. \quad (1)$$

If, in the development of e^{ui} , u be changed to $-u$, $\cos u$ will remain unchanged, because its terms contain only even powers of u ; but $\sin u$ will become $-\sin u$, because its terms contain only odd powers of u . Therefore,

$$e^{-ui} = \cos u - i \sin u. \quad (2)$$

The product of equations (1) and (2) is

$$e^{ui} \times e^{-ui} = (\cos u)^2 - i^2 (\sin u)^2,$$

$$\text{that is,} \quad 1 = (\cos u)^2 - i^2 (\sin u)^2, \quad (\S 255)$$

$$\text{or} \quad 1 = (\cos u)^2 + (\sin u)^2. \quad (\S 531)$$

It is customary to omit the parentheses, and to write this equation as follows :

$$\cos^2 u + \sin^2 u = 1. \quad (3)$$

551. If nu is substituted for u , and if e^{nu} is developed as e^{ni} has been developed, the result will obviously be

$$e^{nu} = \cos nu + i \sin nu.$$

But equation (1) raised to the n th power becomes

$$e^{nu} = (\cos u + i \sin u)^n.$$

Therefore,

$$\cos nu + i \sin nu = (\cos u + i \sin u)^n. \quad (4)$$

If, in this equation $n=2$, the resulting equation is

$$\cos 2u + i \sin 2u = \cos^2 u - \sin^2 u + 2i \sin u \cos u;$$

whence it follows, from § 542, that,

$$\cos 2u = \cos^2 u - \sin^2 u, \quad (5)$$

$$\sin 2u = 2 \sin u \cos u. \quad (6)$$

By substituting $x, y, x+y$, successively, for u in equation (1),

$$e^{xi} = \cos x + i \sin x, \quad (7)$$

$$e^{yi} = \cos y + i \sin y, \quad (8)$$

$$e^{(x+y)i} = \cos (x+y) + i \sin (x+y). \quad (9)$$

The product of equations (7) and (8) is

$$e^{(x+y)i} = \cos x \cos y - \sin x \sin y + i(\sin x \cos y + \cos x \sin y).$$

Therefore,

$$\cos (x+y) + i \sin (x+y) = \cos x \cos y - \sin x \sin y + i(\sin x \cos y + \cos x \sin y);$$

whence (§ 542),

$$\cos (x+y) = \cos x \cos y - \sin x \sin y, \quad (10)$$

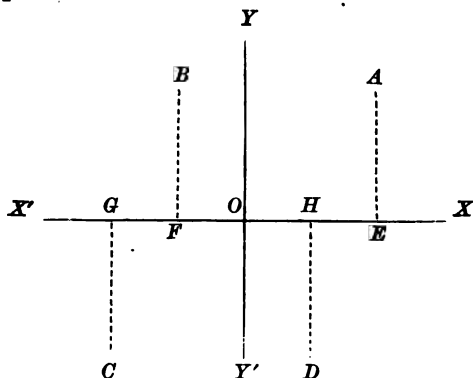
$$\sin (x+y) = \sin x \cos y + \cos x \sin y. \quad (11)$$

CHAPTER XXXII.

LOCI OF EQUATIONS.

552. It is possible to represent equations by diagrams, some of which are regular geometrical figures.

For this purpose, the lines $X'X$ and YY' are drawn perpendicular to each other, intersecting at the point O . These lines may be of any length in drawing the diagram of an equation.



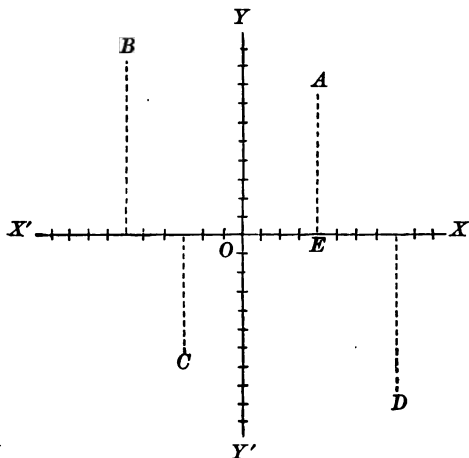
$X'X$ and YY' are called the **Axes of Reference**; $X'X$ is the **Axis of Abscissas**, and YY' the **Axis of Ordinates**.

In order to determine the position of the point A , with reference to these axes, AE is drawn perpendicular to $X'X$, and is called the **Ordinate** of the point A . The line OE from the point O to the foot of the ordinate is the **Abscissa** of A . OE and AE are the **Co-ordinates** of A . What are the co-ordinates of B , C , and D ?

The abscissa of a point is usually designated by the letter x , and its ordinate by y . For the point A , $x = OE$, and $y = AE$.

O is the **Origin of Co-ordinates**, or simply the **Origin**.

553. In representing equations whose roots are arithmetical numbers, numerical values are assigned to the co-ordinates. All ordinates *above* $X'X$ are considered *positive*, and all *below*, *negative*. Abscissas drawn to the *right* of YY' are *positive*, and those to the *left* are *negative*.



Equal lengths of any convenient magnitude are measured on the axes from O , and each length represents a unit. To locate the point whose abscissa is 4 and whose ordinate is 7, a distance of 4 units is measured on $X'X$ from O to the right, and at the point reached a perpendicular 7 units long is erected. A , the extremity of this perpendicular, is the required point.

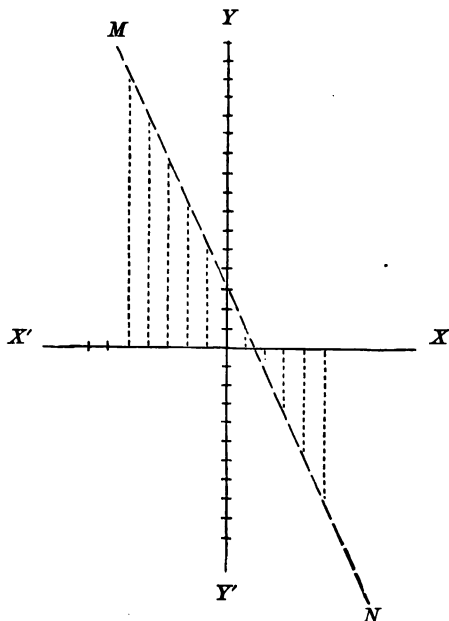
The co-ordinates of B are $x = -6$, and $y = 9$; of C , $x = -3$, and $y = -6$; of D , $x = 8$, and $y = -8$.

554. The equation $2x + y = 3$ may be satisfied by an infinite number of corresponding values of x and y . By changing the equation to the form $y = 3 - 2x$, the following table is readily computed :

If $x=1$, $y= 1$.	If $x=-1$, $y= 5$.
" $x=2$, $y=-1$.	" $x=-2$, $y= 7$.
" $x=3$, $y=-3$.	" $x=-3$, $y= 9$.
" $x=4$, $y=-5$.	" $x=-4$, $y=11$.
" $x=5$, $y=-7$.	" $x=-5$, $y=13$.

This table contains the co-ordinates of ten points. If the points are located with reference to the axes, and the line MN is drawn through them, the line MN is the **Locus** of the given equation.

If MN be prolonged, the value of y which corresponds to any given value of x may be found by laying off an abscissa equal to the given value of x , erecting at its extremity an ordinate terminated by MN , and measuring the length of the ordinate.



If $x=0$, the diagram shows that $y=3$; and if $x=1.5$, $y=0$. If $x=-3.5$, what is the value of y ?

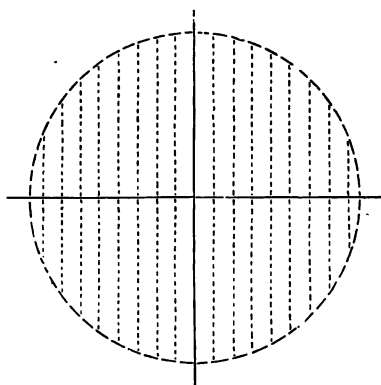
555. In plotting any equation containing x and y , we assume values of x , compute the corresponding values of y , locate the points whose co-ordinates are thus found, and draw the locus through them. When the equation contains only the first powers of x and y , and is of the first degree, the locus is always a straight line. In this case it is necessary to locate only two points of the line; the locus may then be drawn through these points.

556. Let it be required to draw the locus of

$$x^2 + y^2 = 72.25.$$

$$y = \pm \sqrt{72.25 - x^2}.$$

From this equation, by assuming values of x and computing the values of y , the following table may be formed :



If x is	y is
± 1	$\pm 8.44 +$
± 2	$\pm 8.26 +$
± 3	$\pm 7.95 +$
± 4	± 7.50
± 5	$\pm 6.87 +$
± 6	$\pm 6.02 +$
± 7	$\pm 4.82 +$
± 8	$\pm 2.87 +$

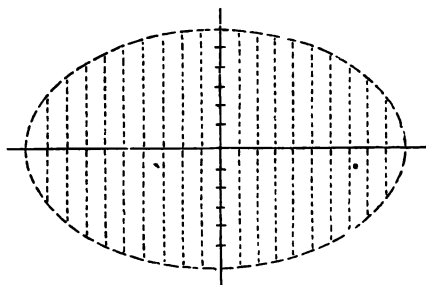
If a curve is drawn through the extremities of the ordinates, the locus is found to be a circle whose centre is at the origin. The diagram shows that x does not reach the value 9. If this value be substituted in the equation, y will be imaginary.

557. Construct the locus of $9x^2 + 25y^2 = 900$.

$$y = \pm \frac{1}{5} \sqrt{900 - 9x^2}$$

From this equation the following table may be formed :

If x is	y is
± 1	$\pm 5.96 +$
± 2	$\pm 5.87 +$
± 3	$\pm 5.72 +$
± 4	$\pm 5.49 +$
± 5	$\pm 5.19 +$
± 6	± 4.80
± 7	$\pm 4.28 +$
± 8	± 3.60
± 9	$\pm 2.61 +$
± 10	0.00
0	$\pm 6.$

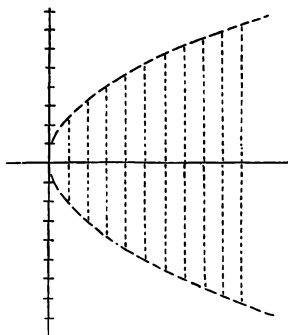


The locus is an ellipse.

558. Construct the locus of $y^2 = 5x$.

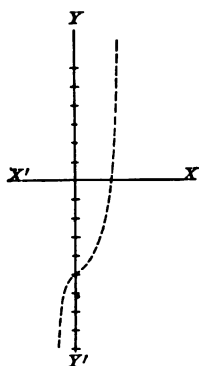
In this case $y = \pm \sqrt{5x}$.

If x is	y is	If x is	y is
1	$\pm 2.23 +$	6	$\pm 5.47 +$
2	$\pm 3.16 +$	7	$\pm 5.91 +$
3	$\pm 3.87 +$	8	$\pm 6.32 +$
4	$\pm 4.47 +$	9	$\pm 6.70 +$
5	± 5.00	10	$\pm 7.07 +$



For negative values of x the ordinates are imaginary. The curve is a parabola; and as x and y may have infinite values, the curve will extend to an infinite distance.

559. Construct the locus of $y = x^3 - x^2 + x - 5$.



If x is	y is	If x is	y is
0.5	-4.625	2.5	+6.875
1.0	-4.000	0.0	-5.000
1.5	-2.375	-0.5	-5.875
2.0	+1.000	-1.5	-12.125

This locus is not a regular geometrical figure.

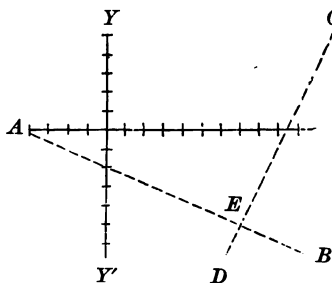
Computation shows that, if $y = 0$, $x = 1.88 +$. Does this result agree with the figure? What is x when y is $+5$? When y is $+10$? When y is -15 ?

560. When two simultaneous equations containing x and y are given, the values of x and y which satisfy both are shown by constructing the loci of both equations with reference to the same axes. In the equations

$$\frac{3x+7y}{2} = -7 \quad \text{and} \quad \frac{2x-y}{5} = 3\frac{1}{2}$$

$$y = -(2 + \frac{3}{7}x), \quad (1); \quad \text{and} \quad y = 2x - 19. \quad (2)$$

AB is the locus of equation (1), and CD is the locus of equation (2). The values of the co-ordinates of any point on AB satisfy (1), and those of any point on CD satisfy (2). Since the point E lies on both AB and CD , its co-ordinates satisfy both equations. By measurement, they are found to be $x = 7$, and $y = -5$. The correctness of these values is easily proved by substituting them



in the given equations. Errors in drawing the diagram will of course affect the accuracy of the results.

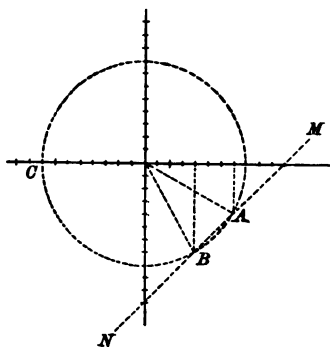
561. In the equations

$$x^2 + y^2 = 65, \quad (1)$$

$$x - y = 11, \quad (2)$$

the circle $ABCD$ is the locus of equation (1), and the line MN is the locus of equation (2). The co-ordinates of A and B satisfy both equations, since A and B lie on both loci. The co-ordinates of A are

$x=7, y=-4$; the co-ordinates of B are $x=4, y=-7$.



Since the square on the hypotenuse of a right triangle equals the sum of the squares on the other two sides, inspection of the diagram shows that (if r is the radius of the circle),

$$r^2 = x^2 + y^2.$$

Hence, the locus of any equation of the form $x^2 + y^2 = r^2$ is a circle whose radius is the square root of the right-hand member. In the present example

$$r = \pm \sqrt{65} = \pm 8.06....$$

EXERCISE 131.

Solve the following equations by constructing their loci:

$$\left. \begin{array}{l} 1. \ 2x + 3y = 8 \\ \quad 3x + 7y = 7 \end{array} \right\} \quad \left. \begin{array}{l} 3. \ 2x - 9y = 11 \\ \quad 3x - 12y = 15 \end{array} \right\}$$

$$\left. \begin{array}{l} 2. \ 3x - 5y = 2 \\ \quad 2x + 7y = 22 \end{array} \right\} \quad \left. \begin{array}{l} 4. \ 4x - 2y = 20 \\ \quad 6x = 9y \end{array} \right\}$$

$$\begin{cases} 5. & 2x - 3y = 4 \\ & 3x + 2y = 32 \end{cases}$$

$$\begin{cases} 6. & 2x + 3y = 7 \\ & 4x - 5y = 3 \end{cases}$$

$$\begin{cases} 7. & 2x - 9y = 11 \\ & 3x - 4y = 7 \end{cases}$$

$$\begin{cases} 8. & 3x - 4y = -5 \\ & 4x - 5y = 1 \end{cases}$$

$$\begin{cases} 9. & x - 2y = 4 \\ & 2x - y = 5 \end{cases}$$

$$\begin{cases} 10. & \frac{3}{x} - \frac{4}{y} = 5 \\ & \frac{4}{x} - \frac{5}{y} = 6 \end{cases}$$

$$\begin{cases} 11. & \frac{1}{x} + \frac{2}{y} = 4 \\ & \frac{3}{x} - \frac{2}{y} = 4 \end{cases}$$

$$\begin{cases} 12. & x^2 + y^2 = 104 \\ & x + y = 12 \end{cases}$$

$$\begin{cases} 13. & x - y = 10 \\ & x^2 + y^2 = 178 \end{cases}$$

$$\begin{cases} 14. & xy - 12 = 0 \\ & x - 2y = 5 \end{cases}$$

$$\begin{cases} 15. & x + y = 13 \\ & xy = 36 \end{cases}$$

$$\begin{cases} 16. & 3y^2 - 4x^2 = 12 \\ & 2x + y = -10 \end{cases}$$

$$\begin{cases} 17. & \frac{4}{5+y} = \frac{5}{12+x} \\ & 2x + 5y = 35 \end{cases}$$

$$\begin{cases} 18. & \frac{2}{x} - \frac{5}{3y} = \frac{4}{27} \\ & \frac{1}{4x} + \frac{1}{y} = \frac{11}{72} \end{cases}$$

562. When a single equation is given containing only one unknown number, all the terms must be transposed to the first member. Thus,

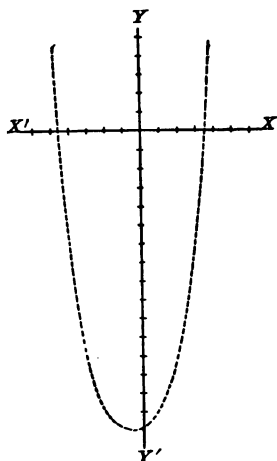
$$\begin{aligned} & x^2 + x - 15.75 = 0; \\ \text{or, if } y=0, & \quad x^2 + x - 15.75 = y. \end{aligned}$$

Assuming values of x , we compute the corresponding values of y , and construct the locus. Now, any value of x which makes $y=0$ satisfies the equation, and is a root; hence, any abscissa whose corresponding ordinate is zero represents a root. The roots, therefore, may be found by measuring the abscissas of the points where the locus meets $X'X$, for at these points $y=0$.

From the given equation the following table may be formed:

If x is	y is	If x is	y is
0	-15.75	-1	-15.75
1	-13.75	-2	-13.75
2	-9.74	-3	-9.75
3	-3.75	-4	-3.75
4	+4.25	-5	+4.25

The table shows that one root is between 3 and 4 (since y changes from - to +, and therefore passes through zero); and, for a like reason, the other is between -4 and -5. The diagram shows that the roots are 3.5 and -4.5.



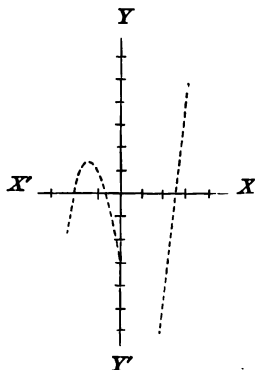
563. Construct the locus of $x^3 - 5x - 3 = 0$.

From this equation the following table may be formed:

x	y	x	y
0	-3	-1	+1
1	-7	-2	-1
2	-5	-3	-15
3	+9		

To plot this locus *accurately*, the values of y must be computed for $x = -1.1, -1.2, -1.3$, etc.

The necessary portions of the locus are given in the diagram, and the roots are found to be 2.5-, -0.6+, and -1.8+; nearer values are 2.49+, -0.65+, and -1.83+.



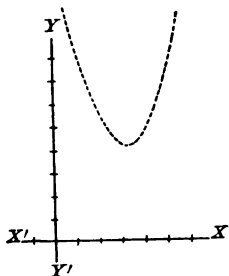
An equation of any degree may be thus plotted, and the locus will be found to cross the axis $X'X$ as many times as the number of *real* roots in the equation.

564. When an equation has no real roots, the locus does not meet $X'X$.

In the equation

$$x^2 - 6x + 13 = 0,$$

both of whose roots are imaginary, the locus, at its nearest approach, is 4 units distant from $X'X$.

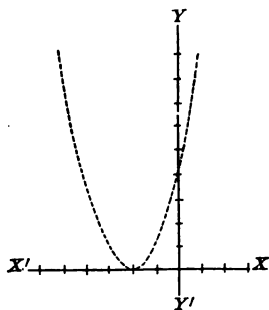


565. The locus of an equation whose roots are equal touches $X'X$, but does not intersect it.

The equation

$$x^2 + 4x + 4 = 0$$

has the roots -2 and -2 , and its locus is shown in the margin.



566. Loci are chiefly valued as *illustrations* of the properties of equations.

EXERCISE 132.

Construct the loci of the following equations :

1. $x^2 + 3x - 10 = 0.$

3. $x^4 - 20x^2 + 64 = 0.$

2. $x^3 - 2x^2 + 1 = 0.$

4. $x^2 - 4x + 10 = 0.$

5. $x^4 - 5x^2 + 4 = 0.$

CHAPTER XXXIII.

EQUATIONS IN GENERAL.

567. EVERY higher equation of which we shall treat can be reduced to the following typical form, which is called the **General Equation of the n th Degree**:

$$x^n + Ax^{n-1} + Bx^{n-2} + Cx^{n-3} + \dots + K = 0. \quad (1)$$

In this the coefficients are real and rational, and the exponents positive integers; the coefficients may be integral or fractional, positive or negative. The equation is arranged according to the descending powers of the unknown number: K is called the **Absolute Term**, because it does not contain x ; but it is classed among the coefficients, since it may be regarded as the coefficient of x^0 . If any one of the coefficients is zero, the corresponding term vanishes.

Reduce to the typical form,

$$6x^4 - 2x^5 + 3x^2 + 20 - 14x = 37.$$

The coefficient of the highest power of x must be made unity. Transposing, arranging, and dividing by -2 , and supplying the missing power of x we have,

$$x^5 - 3x^4 + 0x^3 - \frac{3}{2}x^2 + 7x + \frac{1}{2} = 0.$$

Here $A = -3$, $B = 0$, $C = -\frac{3}{2}$, $D = 7$, $K = \frac{1}{2}$, and $n = 5$.

An **Algebraic Function** of x is an algebraic expression whose value depends upon that of x . A function of x is denoted by such symbols as $F(x)$, $f(x)$, and $\phi(x)$. Thus,

$$7x^3 + 4x^2 + 2x - 5$$

is a function of x .

We shall designate equation (1) by $F(x)=0$; and when its coefficients as well as exponents are all integral, we shall denote it by $f(x)=0$.

568. A Cubic Equation is one of the third degree; as,

$$x^3 - 5x^2 + 7 = 0.$$

A Biquadratic Equation is one of the fourth degree; as,

$$x^4 - 3x^3 + 4x - 15 = 0.$$

A **Root** of an equation is an *expression*, which, when substituted for the unknown number, will satisfy the equation. If the equation is in the typical form, the first member will become zero.

Imaginary roots are found in Higher equations as well as in quadratics.

If an equation having *literal coefficients* is lower than the fifth degree, it can be solved by methods discovered by Ampère, Clausen, Descartes, Euler, and others. Abel has proved that such an equation cannot be solved if its degree is higher than the fourth.

Equations of any degree can be solved *if the coefficients are numerical*.

569. $F(x)$ is exactly divisible by $x-a$, if a is a root of $F(x)=0$.

For the division can be continued until the remainder does not contain x . Let Q represent the quotient, and R the remainder. Since the dividend equals the product of the divisor by the quotient, added to the remainder,

$$(x-a) Q + R = 0.$$

But, by supposition, $x=a$; hence, $(x-a) Q=0$, and so $R=0$. Therefore, the division is exact.

570. Conversely, if $x-a$ is an exact divisor of $F(x)$, a is a root of $F(x)=0$.

But the operation may be still further abridged. As the first term of the divisor is unity, the first term of each remainder is the next term of the quotient; also, the product of the first term of each remainder by the second term of the divisor gives the second term of the remainder. The quotient will not be altered if we use $1+4$ as a divisor, and substitute addition for subtraction in the foregoing process. Thus we have, omitting the first coefficient,

$$\begin{array}{r} -11 + 28 + 2 - 16 + 32 \\ \underline{4 - 28 + 0 + 8 - 32} \\ -7 + 0 + 2 - 8 + 0 \end{array}$$

The last horizontal line contains the coefficients of x in the quotient, except that of the highest power, which is 1. The number 0 is the remainder which comes from the division.

If the coefficients are denoted by A, B, C , etc., and the trial root by r , we have the following rule:

Write A, B, C, \dots, K in a horizontal line.

Under A write r , and write their sum under r .

Multiply the sum by r , and add the product to B ; multiply this sum by r , and add the product to C ; and so on. If the last sum is zero, the division is exact.

This method is called **Synthetic Division**.

When the coefficient of the first term of the dividend is not unity, it must be multiplied by r , and the product added to A ; this sum multiplied by r is then added to B ; and so on.

EXERCISE 133.

Determine whether the number placed in a parenthesis after each equation is a root of the equation:

1. $x^5 + 6x^4 - 10x^3 - 112x^2 - 207x - 110 = 0$. (-5)

2. $x^5 - 8x^4 + 7x^3 + x^2 - 3x + 2 = 0$. (1)

$$3. \quad x^4 + 21x + 7x^3 + 147 = 0. \quad (-7.)$$

$$4. \quad x^5 + 8x^4 - 7x^2 - 54x + 16 = 0. \quad (-8.)$$

$$5. \quad x^4 - 4x^3 - 3x^2 - 2x - 8 = 0. \quad (2.)$$

$$6. \quad x^3 + 14x^2 + 65x + 112 = 0. \quad (-7.)$$

$$7. \quad 2x^4 - 4x^3 - 62x^2 + 114x - 180 = 0. \quad (6.)$$

$$8. \quad x^4 - 7x - 2x^2 - 15 = 0. \quad (-5.)$$

$$9. \quad x^4 + 2.3x^3 + 3.6x^2 + 4.9x + 1.2 = 0. \quad (-0.3.)$$

$$10. \quad x^3 - \frac{1}{8}x^2 - \frac{1}{12}x - \frac{1}{8} = 0. \quad (\frac{2}{3}.)$$

573. If $F(x) = (x - a)(x - b)(x - c)$, etc., a, b, c , etc., are the roots of $F(x) = 0$.

For, by hypothesis,

$$(x - a)(x - b)(x - c)(x - d) \dots = 0.$$

This equation is satisfied if $x = a$, for this supposition will reduce the first member to zero. The equation will be satisfied if $x = a$, or if $x = b$, or if $x = c$, or if $x = d$, etc.; so, by definition, a, b, c, d , etc., are roots of $F(x) = 0$.

The degree of the equation equals the number of binomial factors. When there are n factors, the equation is of the n th degree; conversely, when the equation is of the n th degree, and resolvable into factors of the form $(x - a)$, there are n of those factors.

574. When $F(x) = 0$ is of the n th degree, it has n roots and no more.*

The equation is, § 567,

$$x^n + Ax^{n-1} + Bx^{n-2} + Cx^{n-3} + \dots + K = 0. \quad (1)$$

* **NOTE.** It is assumed that every equation has at least one root. The demonstration of this fact, which is long and difficult, need not be given here.

Let a be a root of equation (1); then, by § 569, the equation is divisible by $x - a$, and the quotient will be of the form

$$x^{n-1} + A'x^{n-2} + B'x^{n-3} + C'x^{n-4} + \dots + K' = 0. \quad (2)$$

Now (2) must have at least one root, and let b be that root. Dividing (2) by $(x - b)$, dividing the resulting equation by $(x - c)$, c being a root of that equation, and continuing the process until $n - 1$ divisions have been made, we reach a result of the form

$$x - f = 0. \quad (3)$$

It is plain, therefore, that the first member of equation (1) is the product of n factors of the form $(x - a)$, and so, by § 573, it has n roots. If there were more than n roots there would be more than n factors, and the degree of the equation would be higher than the n th. These roots are not necessarily all different. The roots of

$$x^3 - x^2 - 8x + 12 = 0$$

are 2, 2, and -3 .

This equation has two *equal roots*.

575. When the roots of an equation are given, the equation is readily found.

Find the equation whose roots are 3, 7, -1 , and -2 .

$$(x - 3)(x - 7)(x + 1)(x + 2) = 0;$$

or, by multiplying,

$$x^4 - 7x^3 - 7x^2 + 43x + 42 = 0.$$

EXERCISE 134.

Find the equations whose roots are given below:

1. 2, 6, and -7 .
2. 1, 4, -1 , and -3 .
3. 2, 3, -2 , -3 , and -6 .
4. 0.2 , $\frac{1}{8}$, and -0.4 .
5. 5 , $3 + \sqrt{-1}$, and $3 - \sqrt{-1}$.

576. The relation between the coefficients and the roots of an equation is to be investigated.

According to § 575, a and b are the roots of

$$x^2 - (a + b)x + ab = 0; \quad (1)$$

a , b , and c are the roots of

$$x^3 - (a + b + c)x^2 + (ab + ac + bc)x - abc = 0; \quad (2)$$

a , b , c , and d are the roots of

$$x^4 - (a + b + c + d)x^3 + (ab + ac + ad + bc + bd + cd)x^2 - (abc + abd + acd + bcd)x + abcd = 0; \quad (3)$$

and so on.

From these the following conclusions may be drawn:

I. The coefficient of the second term, with its sign changed, equals the sum of the roots; the coefficient of the fourth term, with its sign changed, equals the sum of the different products that can be formed by taking the roots in groups of three; and so on.

II. The coefficient of the third term equals the sum of the different products that can be formed by taking the roots in groups of two; the coefficient of the fifth term equals the sum of the different products that can be formed by taking the roots in groups of four; and so on.

III. The absolute term equals the continued product of the roots with their signs changed.

EXERCISE 135.

By the above principles, form the equations whose roots are given below:

1. 2, 4, and -3 .

4. 6, 6, and 6.

2. 2, -1 , and -7 .

5. 2, 1, -2 , and -1 .

3. 2, 0, and -2 .

6. 2, $\frac{1}{2}$, -2 , and $-\frac{1}{2}$.

SOLUTIONS BY FACTORING.

577. $F(x)$ is sometimes factored by inspection, and the roots of $F(x)=0$ are thus found:

(1) Solve $x^2 - 7x - 18 = 0$.

By factoring, we have $(x-9)(x+2) = 0$.

Therefore, by § 570, $x = 9$, or -2 .

(2) Solve $x^3 - 8 = 0$.

$$(x-2)(x^2 + 2x + 4) = 0. \quad (1)$$

Now, as equation (1) may be satisfied by putting either factor equal to zero, we have

$$x - 2 = 0, \quad (2)$$

and $x^2 + 2x + 4 = 0. \quad (3)$

From (2) we see that $x = 2$, and the solution of (3) gives

$$x = -1 + \sqrt{-3}, \text{ or } -1 - \sqrt{-3}.$$

(3) Solve $4x^4 - x^2 + 2x - 1 = 0$.

$$4x^4 - (x^2 - 2x + 1) = 0,$$

$$(2x^2)^2 - (x-1)^2 = 0,$$

$$(2x^2 + x - 1)(2x^2 - x + 1) = 0.$$

$$2x^2 + x - 1 = 0. \quad (1)$$

$$2x^2 - x + 1 = 0. \quad (2)$$

From (1), $x = \frac{1}{2}$ or -1 ;

from (2) $x = \frac{1}{2}(1 + \sqrt{-7})$ or $\frac{1}{2}(1 - \sqrt{-7})$.

EXERCISE 136.

Find the roots of the following:

1. $x^2 + 11x + 24 = 0$.

2. $7x^2 + 161x + 714 = 0$. (Divide by 7.)

3. $x^4 - 4a^2x^2 + 3a^4 = 0$.

4. $x^5 + 4x^3 + 8x^2 + 32 = 0$. (Divide by $x^3 + 8$.)

5. $12x^2 - 5x - 2 = 0$.

6. $4x^4 - 9x^3 + 6x - 1 = 0$. 8. $x^6 - 64 = 0$.
 7. $49x^3 - 112bx + 64b^2 = 0$. 9. $3x^3 - x^2 + 3x - 1 = 0$.
 10. $x - 27x^4 = 0$.

578. Cubic equations in which at least one root is integral are easily factored. Let $-a$ be an integral root; then the equation may be written in the form

$$(x + a)(x^2 + mx + n) = 0, \quad (1)$$

or $x^3 + (a + m)x^2 + (n + am)x + an = 0. \quad (2)$

$$\begin{array}{r} a \\ \hline m \end{array} \qquad \begin{array}{r} n \\ \hline am \end{array} \qquad am \div m = a.$$

By trial, a , m , and n are to be obtained. Resolve the absolute term into two trial factors. If these factors are the correct values of a and n , when a is subtracted from the coefficient of x^2 , and n from the coefficient of x , as above, the second remainder divided by the first remainder will give the first subtrahend, as shown above. In this example, a is the first subtrahend, m the first remainder, and n the second subtrahend; $-a$ is one root, and by solving $x^2 + mx + n = 0$, the other roots are determined.

Solve $x^3 - 9x^2 + 26x - 24 = 0$.

$$\begin{array}{r} x^3 - 9x^2 + 26x - 24 = 0 \\ -3 \quad + \quad 8 \\ \hline -6 \quad + \quad 18 \end{array} \qquad 18 \div -6 = -3.$$

$\therefore x - 3 = 0$, and $x^2 - 6x + 8 = 0$; whence, $x = 3, 2$, and 4 .

EXERCISE 137.

Solve the following:

1. $x^3 + 3x^2 - 25x - 12 = 0$. 6. $x^3 - 3x^2 - 54x - 104 = 0$.
 2. $x^3 - 4x^2 - 8x + 8 = 0$. 7. $x^3 + 9x^2 + 2x - 48 = 0$.
 3. $x^3 - 7x^2 + 19x - 21 = 0$. 8. $x^3 - 2x^2 - 25x + 50 = 0$.
 4. $x^3 - 8x^2 + 21x - 18 = 0$. 9. $x^3 - 3x^2 - 61x + 63 = 0$.
 5. $x^3 - 26x - 5 = 0$. 10. $x^3 - 37x - 84 = 0$.

579. Biquadratic equations can be solved if they can be resolved into two factors of the second degree, even when no root is integral.

$$(x^2 + mx + n)(x^2 + px + q)$$

$$= x^4 + (m+p)x^3 + (n+mp+q)x^2 + (np+mq)x + nq.$$

$$\text{Solve } x^4 + 13x^3 + 33x^2 + 31x + 10 = 0.$$

The following are the values of the coefficients :

$$m + p = 13, \quad (1)$$

$$n + mp + q = 33, \quad (2)$$

$$np + mq = 31, \quad (3)$$

$$\text{and} \quad nq = 10. \quad (4)$$

Take trial-values of n and q from (4). From (2) find the value of mp . Combine this with (1), and get by inspection the values of m and p . If the values of m , n , p , and q , satisfy (3), they are correct.

FIRST HYPOTHESIS.

$$\text{Let} \quad n = -2, \text{ and } q = -5.$$

$$\text{Then, from (2),} \quad mp = 40,$$

$$\text{and, using (1),} \quad m = 5, \text{ and } p = 8.$$

These values will not satisfy (3).

SECOND HYPOTHESIS.

$$\text{Let} \quad n = 1, \text{ and } q = 10.$$

$$\text{Then, from (2),} \quad mp = 22,$$

$$\text{and, using (1),} \quad m = 2, \text{ and } p = 11.$$

These values satisfy (3). Since the two factors of the biquadratic are $(x^2 + mx + n)$ and $(x^2 + px + q)$, each of which equals zero, we have, in this example,

$$x^2 + 2x + 1 = 0,$$

$$\text{and} \quad x^2 + 11x + 10 = 0;$$

the solution of which gives the values of x .

The values of n and q can be interchanged without affecting the values of m and p . If any hypothesis makes m and p contain surds, it is to be rejected, unless $n = q$; for, if m and p contain surds, $(np + mq)$, the coefficient of x , will contain surds, unless $n = q$; in which case the surds may disappear.

EXERCISE 138.

Solve the following :

1. $x^4 - 2x^3 - 13x^2 + 38x - 24 = 0.$
2. $x^4 - 5x^3 - 2x^2 + 12x + 8 = 0.$
3. $x^4 - 4x^3 - 8x + 32 = 0.$
4. $x^4 - 12x^3 + 50x^2 - 84x + 49 = 0.$
5. $x^4 - 11x^2 + 18x - 8 = 0.$
6. $x^4 - 10x^2 - 20x - 16 = 0.$
7. $x^4 - 7x^3 + 23x^2 - 47x + 42 = 0.$
8. $x^4 + 2x^3 - 9x^2 - 8x + 20 = 0.$
9. $x^4 - 4x^3 - 102x^2 - 188x - 91 = 0.$
10. $x^4 - 11x^3 + 46x^2 - 117x + 45 = 0.$

DESCARTES' RULE OF SIGNS.

580. A **Complete Equation** is one in which no power of x is wanting.

If two successive terms have like signs, there is a *permanence* of sign; if they have unlike signs, there is a *variation*.

Thus, in the complete equation

$$x^7 - x^6 - 5x^5 + 4x^4 + 2x^3 - 3x^2 + 6x + 7 = 0,$$

there are three permanences and four variations.

Descartes discovered that *the signs of the roots are related to the signs of the terms* of an equation.

581. Descartes' Rule (modified) is as follows :

No complete equation has a greater number of positive roots than of variations of sign, nor a greater number of negative roots than of permanences of sign.

Suppose that the successive terms of a certain complete equation have the signs

$$+ - + + + - - +$$

Here are three permanences and four variations: To introduce a new positive root into the equation, we multiply it by $x - a$. The resulting signs are shown in the following scheme :

$$\begin{array}{cccccccc}
 + & - & + & + & + & - & - & + \\
 + & - & & & & & & \\
 \hline
 + & - & + & + & + & - & - & + \\
 & & - & + & - & - & - & + & + & - \\
 \hline
 + & - & + & \pm & \pm & - & \pm & + & -
 \end{array}$$

The sign \pm indicates that there is doubt whether the term is positive or negative. Inspection shows that where there were permanences in the multiplicand, there are ambiguities in the product. Therefore, the introduction of the root $+a$ has not increased the number of permanences. But, as it has made the number of terms one greater, the number of variations must have been increased by *one, at least*.

As the introduction of a positive root increases the number of variations, the whole number of positive roots cannot exceed the number of variations. The root $-a$ can be introduced into an equation by multiplying it by $x + a$; and, by reasoning similar to the preceding, it can be shown that the number of negative roots cannot exceed the number of permanences.

582. *When all the roots of a complete equation are real, the number of positive roots equals the number of variations, and the number of negative roots equals the number of permanences.*

Let p be the number of positive roots, n the number of negative roots, v' the number of variations, and p' the number of permanences.

Now the degree of the equation equals $p+n$, and also equals $v'+p'$. Therefore,

$$p+n=v'+p';$$

or, by transposing, $v'-p=n-p'$. (1)

Now, by § 581, the first member of equation (1) must be positive, or zero; also the second member must be negative, or zero. Since the first member equals the second, both are zero. Hence, $v'=p$, and $n=p'$.

583. In an equation which lacks one power of x , the presence of imaginary roots may sometimes be detected. For illustration, take the equation

$$x^3 \pm 0x^2 + 5x + 7 = 0.$$

We are at liberty to assume that the second term is positive, or that it is negative.

Taking it as positive, there is no variation, and the equation has, therefore, no positive root.

Assuming the second term to be negative, there is only one permanence, and so there cannot be more than one negative root.

As there are three roots, and as imaginary roots come in pairs (§ 586), we conclude that one root is negative, and two are imaginary.

584. A complete equation whose signs are all positive can have no positive real root, for there are no variations of sign. When the signs are alternately positive and negative, there are no negative real roots, for there are no permanences of sign.

EXERCISE 139.

All the roots of the equations given below are real; determine their signs.

1. $x^4 + 4x^3 - 43x^2 - 58x + 240 = 0$.
2. $x^3 - 22x^2 + 155x - 350 = 0$.
3. $x^4 + 4x^3 - 35x^2 - 78x + 360 = 0$.
4. $x^3 - 12x^2 - 43x - 30 = 0$.
5. $x^5 - 3x^4 - 5x^3 + 15x^2 + 4x - 12 = 0$.
6. $x^3 - 12x^2 + 47x - 60 = 0$.
7. $x^4 - 2x^3 - 13x^2 + 38x - 24 = 0$.
8. $x^5 - x^4 - 187x^3 - 359x^2 + 186x + 360 = 0$.
9. $x^6 - 10x^5 + 19x^4 + 110x^3 - 536x^2 + 800x - 384 = 0$.
10. $x^7 - 10x^6 + 22x^5 + 32x^4 - 131x^3 + 50x^2 + 108x - 72 = 0$.

FRACTIONAL AND IMAGINARY ROOTS.

585. *A rational fraction is never a root of $f(x) = 0$.*

Suppose that $\frac{l}{t}$, a simple fraction in its lowest terms, is a root of

$$x^n + Ax^{n-1} + Bx^{n-2} + Cx^{n-3} + \dots + K = 0. \quad (1)$$

Substituting $\frac{l}{t}$ for x , and multiplying by t^{n-1} , we obtain

$$\frac{l^n}{t} + Al^{n-1} + Btl^{n-2} + Ct^2l^{n-3} + \dots + Kt^{n-1} = 0. \quad (2)$$

But the sum of a fraction and a number of integers cannot equal zero. Therefore, equation (2) is false, and $\frac{l}{t}$ is not a root of equation (1).

586. *Imaginary roots enter equations in conjugate pairs.*

Any imaginary root may be represented by the expression $a \pm bi$. (See § 538, and the last sentences of §§ 541 and 546.) When a is zero, the root is a pure imaginary.

There must be an *even* number of imaginary roots, or the absolute term, which is the product of all the roots with their signs changed, would be imaginary; hence these roots are in pairs.

The product of $x - (a + bi)$ by $x - (a - bi)$ is real. If the roots could not be grouped in conjugate pairs, the product $(x - a)(x - b)(x - c) \dots$ would involve imaginaries.

587. *If $F(x) = 0$ is of an odd degree, it has at least one real root; if of an even degree, all the roots may be imaginary.*

An equation of an odd degree has an odd number of roots; hence, all the roots cannot be imaginary. § 586.

When the equation is of an even degree, it can be resolved into an even number of factors, as follows:

$$(x - a)(x - b)(x - c)(x - d)(x - e)(x - f) \dots = 0. \quad (1)$$

If $(x - a)(x - b) = x^2 + px + q,$

and $(x - c)(x - d) = x^2 + p_1x + q_1,$ and so on,

equation (1) becomes

$$(x^2 + px + q)(x^2 + p_1x + q_1)(x^2 + p_2x + q_2) \dots = 0.$$

Putting each of these trinomial factors equal to zero, and solving the resulting equations,

if $\frac{p^2}{4} < q,$ and $\frac{p_1^2}{4} < q_1,$ and so on,

all the roots will be imaginary.

EQUAL ROOTS.

588. In finding the equal roots of an equation, expressions called **Derivatives** are used, which play a very important part in the Differential Calculus.

Let x be a variable (§ 467). If it is gradually increasing or diminishing, $3x$ will increase or diminish three times as fast; $7x$, seven times as fast; and mx , m times as fast; m is the ratio of the rate of change in mx to the rate of change in x .

Thus, if $x=3$, $7x=21$. Suppose that x increases from 3.0 to 3.1, then $F(x)$ will increase from 21.0 to 21.7 in the same time. That is,

$$\frac{\text{the increase of } F(x)}{\text{the increase of } x} = \frac{0.7}{0.1} = \frac{7}{1}.$$

But while x increases at a uniform rate, $F(x)$ may not increase at a uniform rate.

Thus, if $F(x)=5x^3-x^2+4x-1$, when $x=1$, $F(x)=7$; when $x=2$, $F(x)=43$; when $x=3$, $F(x)=137$.

Hence, to find the ratio between the increments of $F(x)$ and of x at the instant x begins to change from any particular value, the increment of x must approach indefinitely to zero. In general,

The derivative of $F(x)$ is the limit of the ratio of the increment of $F(x)$ to that of x as the increment of x approaches indefinitely to zero.

An increment may be positive or negative.

The derivative of $F(x)$ may be found as follows:

I. In $F(x)$ substitute $x+h$ for x , h being an increment that approaches zero as a limit. Denote the result by $F(x+h)$. Subtract $F(x)$ from $F(x+h)$: the remainder is the increment of $F(x)$.

II. Divide this remainder, $[F(x+h) - F(x)]$, by h : the quotient is the ratio of the increment of $F(x)$ to that of x .

III. Any term of this quotient that contains h as a factor, will have zero for a limit, and therefore vanish.

589. Find the derivative of $ax^2 + b$.

By I., $F(x) = ax^2 + b$,
and $F(x+h) = ax^2 + 2axh + ah^2 + b$.
 $\therefore F(x+h) - F(x) = 2axh + ah^2$.

By II., $\frac{F(x+h) - F(x)}{h} = 2ax + ah$.

By III., $2ax$ is the derivative of $ax^2 + b$.

Find the derivative of x^2 .

$$(x+h)^2 = x^2 + 2xh + h^2,$$

$$\frac{(x+h)^2 - x^2}{h} = 2x + h.$$

By III., $2x$ is the derivative of x^2 .

Find the derivative of x^3 .

$$\frac{(x+h)^3 - x^3}{h} = 3x^2 + 3xh + h^2.$$

By III., $3x^2$ is the derivative of x^3 .

In general, the derivative of x^n is nx^{n-1} .

For $x^n = x$ taken n times as a factor; and if x increases, the increase of each x is multiplied by the continued product of all the others; that is, by x^{n-1} . But there are n of these increasing quantities, and the derivative is therefore nx^{n-1} .

Hence, the derivative of a *simple expression* is found by multiplying the expression by the exponent of the variable, and diminishing the exponent by unity.

590. When $F(x)$ contains a number of terms, involving different powers of x , the derivative of the entire polynomial is equal to the sum of the derivatives of the separate terms.

For, if I' , I'' , I''' , I'''' , etc., represent the increments of the successive terms of the polynomial (when $x + h$ is substituted for x),

$$\frac{I' + I'' + I''' + I'''' + \dots}{h} + \dots$$

is the derivative of $F(x)$ (when h approaches indefinitely to zero). Likewise,

$$\frac{I'}{h}, \frac{I''}{h}, \frac{I'''}{h}, \frac{I''''}{h}, \dots$$

are the derivatives of the different terms. It is plain that

$$\frac{I' + I'' + I''' + I'''' + \dots}{h} + \dots = \frac{I'}{h} + \frac{I''}{h} + \frac{I'''}{h} + \frac{I''''}{h} + \dots$$

Find the derivative of $4x^3 + 3x^2 + 5x + 7 = 0$.

Let $F(x) = 4x^3 + 3x^2 + 5x + 7$.

By substitution,

$$\begin{aligned} F(x+h) - F(x) &= 12hx^2 + (12h^2 + 6h)x + 4h^3 + 3h^2 + 5h. \\ \therefore \frac{F(x+h) - F(x)}{h} &= 12x^2 + (12h + 6)x + 4h^2 + 3h + 5. \end{aligned}$$

By letting h approach zero as a limit, and denoting the derivative by $F'(x)$,

$$F'(x) = 12x^2 + 6x + 5.$$

But $12x^2$, $6x$, and 5 , are respectively the derivatives of $4x^3$, $3x^2$, and $5x$. By considering the last term of $F(x)$ as the coefficient of x^0 , its derivative would be $7 \times 0x^{-1}$, which equals 0.

The derivative of a given expression is called its **first derivative**; the derivative of the first derivative is called its **second derivative**, etc. The second derivative is denoted by $F''(x)$; the third by $F'''(x)$, etc.

From what has preceded, the following rule for finding the first derivative of $F(x)$ is obtained:

Multiply each term by the exponent of x in it, and diminish the exponent by unity.

591. Find the successive derivations of

$$x^6 + 3x^5 - 2x^4 - 5x^3 + 2x^2 + 7x - 12.$$

$$F^I(x) = 6x^5 + 15x^4 - 8x^3 - 15x^2 + 4x + 7,$$

$$F^{II}(x) = 30x^4 + 60x^3 - 24x^2 - 30x + 4,$$

$$F^{III}(x) = 120x^3 + 180x^2 - 48x - 30,$$

$$F^{IV}(x) = 360x^2 + 360x - 48,$$

$$F^V(x) = 720x + 360,$$

$$F^{VI}(x) = 720,$$

$$F^{VII}(x) = 0.$$

EXERCISE 140.

Find the successive derivatives of the polynomials:

1. $x^2 + 2x + 3.$

3. $x^4 + 2x^3 - 5x^2 + 64.$

2. $x^3 - 3x^2 + 7x + 25.$

4. $x^5 + x^4 - 6x^3 + 3x^2 - 4x + 27.$

5. $x^4 - 3ax^3 + 6bx^2 - 9cx + mn.$

592. *The derivative of $F(x)$ is the coefficient of the first power of h in $F(x + h)$.*

For, $F(x + h)$ is composed of $F(x)$ and terms each of which contains some power of h .

$F(x + h) - F(x)$ is composed of those terms of $F(x + h)$ of which each contains some power of h .

$\frac{F(x + h) - F(x)}{h}$ is composed of the *coefficients* of the first powers of h in $F(x + h)$, and of terms containing h ; rejecting the terms containing h (III., § 588), the derivative of $F(x)$ is the coefficient of the first power of h in $F(x + h)$.

Find the derivative of $F(x) = ax^3 + bx^2 + cx + d.$

$$\begin{aligned} F(x + h) &= ax^3 + 3ax^2h + 3axh^2 + ah^3 \\ &\quad + bx^2 + 2bxh + bh^2 \\ &\quad + cx + ch \\ &\quad + d. \end{aligned}$$

$$F(x + h) - F(x) = (3ax^2 + 2bx + c)h + (3ax + b)h^2 + ah^3,$$

$$\frac{F(x+h) - F(x)}{h} = 3ax^2 + 2bx + c + (3ax + b)h + ah^2.$$

$$\therefore F'(x) = 3ax^2 + 2bx + c.$$

593. Apply these principles to the detection of the equal roots of $F(x) = 0$.

This equation may be written in the form

$$(x-a)(x-b)(x-c)(x-d) \dots = 0. \quad (1)$$

Substituting $x+h$ for x ,

$$(x-a+h)(x-b+h)(x-c+h)(x-d+h) \dots = 0. \quad (2)$$

If (2) be expanded, and powers of h higher than the first be neglected, the h in the first parenthesis will be multiplied by

$$(x-b)(x-c)(x-d) \dots;$$

the h in the second parenthesis will be multiplied by

$$(x-a)(x-c)(x-d) \dots;$$

and so on.

The total coefficient of h will be

$$\left. \begin{aligned} &(x-b)(x-c)(x-d) \dots \\ &+ (x-a)(x-c)(x-d) \dots \\ &+ (x-a)(x-b)(x-d) \dots \end{aligned} \right\} \quad (3)$$

Now (3) is the derivative of $F(x)$, by § 592.

When (1) has two equal roots, let $a=b$. We see that $x-a$ is, in this case, the H. C. F. of (1) and (3). When (1) has three equal roots, let $a=b=c$; then $(x-a)^2$ is the H. C. F. of (1) and (3). In general, when $F(x)=0$ has m roots equal to a , and p roots equal to b , the H. C. F. of $F(x)$ and $F'(x)$ is $(x-a)^{m-1}(x-b)^{p-1}$.

594. Hence, in seeking the equal roots of $F(x)=0$, if $\phi(x)$ represents the H. C. F., it will be seen that:

The roots of $\phi(x)=0$ will each occur once more in $F(x)=0$ than in $\phi(x)=0$.

When there is no H. C. F. there are no equal roots. In some cases, $\phi(x)=0$ is of so high a degree that it cannot

be solved conveniently. If, however, it has equal roots, they may be found as above.

$$\begin{aligned} \text{Solve } x^7 + 4x^6 - 20x^5 - 50x^4 + 175x^3 \\ + 118x^2 - 588x + 360 = 0. \end{aligned} \quad (1)$$

$$\begin{aligned} F'(x) &= 7x^6 + 24x^5 - 100x^4 - 200x^3 + 525x^2 + 236x - 588, \quad (2) \\ \phi(x) &= x^3 - x^2 - 8x + 12, \end{aligned}$$

and the equation to be solved is,

$$x^3 - x^2 - 8x + 12 = 0. \quad (3)$$

Apply § 578 to this case, or else proceed to find the equal roots of (3). The first derivative is

$$3x^2 - 2x - 8. \quad (4)$$

The H. C. F. of (3) and (4) is $x - 2$. Since $x - 2 = 0$, $x = 2$, and the root 2 occurs twice in (3). Dividing (3) by $(x - 2)^2$ gives $x + 3 = 0$; hence, -3 is the other root of (3). As the root 2 occurs twice in (3), and -3 once, we know that (1) has as roots, 2, 2, 2, -3 , and -3 .

By dividing (1) by $(x - 2)^3(x + 3)^2$, the quotient is

$$x^2 + 4x - 5 = 0.$$

The solution of this equation gives $x = 1$ or -5 . The seven roots of (1) are, therefore, 1, 2, 2, 2, -3 , -3 , and -5 .

EXERCISE 141.

Find all the roots of the following :

1. $x^3 - 8x^2 + 13x - 6 = 0$.
2. $x^3 - 7x^2 + 16x - 12 = 0$.
3. $x^4 - 6x^3 - 8x - 3 = 0$.
4. $x^3 - 2x^2 - 15x + 36 = 0$.
5. $x^4 - 7x^3 + 9x^2 + 27x - 54 = 0$.
6. $x^4 - 24x^2 + 64x - 48 = 0$.
7. $x^4 - 10x^3 + 24x^2 + 10x - 25 = 0$.
8. $x^5 - 11x^4 + 19x^3 + 115x^2 - 200x - 500 = 0$.
9. $x^5 - 2x^4 + 3x^3 - 7x^2 + 8x - 3 = 0$.
10. $x^4 + 6x^3 + x^2 - 24x + 16 = 0$.

TRANSFORMATION OF EQUATIONS.

595. The solution of equations is usually facilitated by reducing them to the form $f(x)=0$; and, since we treat only of equations whose exponents are positive integers, we have simply to make the coefficient of the first term unity, and the succeeding coefficients integral. If the exponents were negative or fractional, the equation could still be reduced to the form $f(x)=0$.

The coefficient of the first term may be reduced to unity, without altering the values of the roots, by the following rule:

Divide the equation by the coefficient of the highest power of x .

When this division makes any of the coefficients fractional, the equation is to be transformed into another whose coefficients are integers.

Let m be the L. C. M. of the denominators of the fractions.

Substitute $\frac{y}{m}$ for x in the equation

$$x^n + Ax^{n-1} + Bx^{n-2} + Cx^{n-3} + \dots + K = 0, \quad (1)$$

and clear of fractions; this gives

$$y^n + Amy^{n-1} + Bm^2y^{n-2} + Cm^3y^{n-3} + \dots + Km^n = 0. \quad (2)$$

The coefficients of (2) must be integral; because m is the L. C. M. of the denominators of A , B , C , etc. The roots of (2) divided by m are the roots of (1); for, by hypothesis.

$$x = \frac{y}{m}.$$

Hence, to remove fractional coefficients,

Multiply the coefficient of x^{n-1} by m , that of x^{n-2} by m^2 , etc., the absolute term being multiplied by m^n .

A little study will sometimes enable us to find a number less than the L. C. M., whose successive powers will clear the equation of fractions.

Reduce $3x^3 - 4x^2 + \frac{1}{2}x - \frac{3}{4} = 0$
to the form $f(x) = 0$.

Divide by 3, $x^3 - \frac{4}{3}x^2 + \frac{1}{6}x - \frac{1}{4} = 0$.

The L. C. M. of the denominator is 12; and, by the last rule,

$$x^3 - 16x^2 - 24x - 432 = 0,$$

which is in the form required.

EXERCISE 142.

Put these equations in the form $f(x) = 0$:

1. $2x^3 + \frac{3}{2}x^2 - x + \frac{1}{6} = 0$.
3. $5x^4 - x^3 - \frac{1}{2}x^2 - \frac{1}{3}x + 1 = 0$.
2. $3x^3 + 5x^2 - \frac{1}{2}x - 8 = 0$.
4. $x^5 + \frac{1}{2}x^4 + \frac{3}{8}x^3 - \frac{1}{8}x^2 + x - 3 = 0$.
5. $x^4 - 2x^2 + \frac{1}{2}x - 14 = 0$. (Supply the term $0x^3$.)

Hereafter we shall treat only of equations which are in the form $f(x) = 0$.

596. *The signs of the roots of $f(x) = 0$ are changed by changing the signs of the alternate terms, beginning with the second.*

By § 576, I. and II., it is plain that changing the signs of the roots changes the signs of the alternate terms, beginning with the second, and does not change the signs of the other terms. Conversely, if the signs of the alternate terms of a given equation are changed, beginning with the second term, an equation is obtained whose roots are the roots of the given equation with their signs changed. Missing powers of x must be supplied with the coefficient 0.

597. *The roots of $f(x)=0$ are multiplied by m , by multiplying the second term by m , the third by m^2 , etc.*

In removing fractional coefficients, we multiply x^{n-1} by m , x^{n-2} by m^2 , etc.; and it has been shown, in § 595, that the roots of the resulting equation are m times the roots of the original equation.

598. *To obtain an equation whose roots are the reciprocals of those of $f(x)=0$, write the coefficients in reverse order.*

In the given equation

$$x^n + Ax^{n-1} + Bx^{n-2} + \dots + Ix^2 + Jx + K = 0, \quad (1)$$

substitute $\frac{1}{y}$ for x , multiply by y^n , and change the order of the terms; the result is

$$Ky^n + Jy^{n-1} + Iy^{n-2} + \dots + By^2 + Ay + 1 = 0. \quad (2)$$

Comparison of (1) and (2) shows that the order of the coefficients has been reversed.

By dividing by K , (2) may be reduced to the form $f(x)=0$.

Thus, the equation whose roots are the reciprocals of the roots of

$$x^6 - 5x^5 - 7x^4 + 6x^3 + 13x^2 - 9x + 25 = 0,$$

$$\text{is } 25x^6 - 9x^5 + 13x^4 + 6x^3 - 7x^2 - 5x + 1 = 0.$$

599. In solving some higher numerical equations, it is necessary to transform $f(x)=0$ into an equation whose roots are less by h .

In the equation

$$x^n + Ax^{n-1} + Bx^{n-2} + \dots + Jx + K = 0, \quad (1)$$

by putting $x = y + h$ (in which case $y = x - h$, as required),

$$(y+h)^n + A(y+h)^{n-1} + B(y+h)^{n-2} + \dots + J(y+h) + K = 0. \quad (2)$$

Expanded by the binomial theorem, equation (2) becomes

$$\begin{array}{l} y^n + A \left| y^{n-1} \right. \\ \quad + nh \left| \begin{array}{l} + B \\ + (n-1) Ah \\ + \frac{1}{2} n(n-1) h^2 \end{array} \right. \end{array} y^{n-2} + \dots + K = 0. \quad (3)$$

Let $A + nh = A'$, $B + (n-1)Ah + \frac{1}{2}n(n-1)h^2 = B'$, etc., and $x-h=y$; equation (3) then reduces to the form

$$(x-h)^n + A'(x-h)^{n-1} + B'(x-h)^{n-2} + \dots + J'(x-h) + K' = 0. \quad (4)$$

Divide the first member of (4) by $(x-h)$, and the remainder is K' ; divide the quotient just found by $(x-h)$, and the remainder is J' . By continuing the divisions, all the coefficients of (4) can be found.

The values of x are the same in (4) and (1); therefore, the first member of (4), when developed, equals the first member of (1). Hence, if the successive divisions were performed upon (1), the remainders would be the same.

600. The rule, therefore, for transforming $f(x)=0$ into an equation whose roots are less by h , is as follows:

Divide $f(x)$ by $x-h$, and the remainder will be the absolute term of the transformed equation. Divide the quotient just found by $x-h$, and the remainder will be the coefficient of the last term but one of the transformed equation. Continue the process until all the coefficients are determined.

By using $x+h$ as the divisor, $f(x)=0$ can be transformed into an equation whose roots are greater by h .

The above rule will apply when the coefficient of x^n is not unity. It is best to employ Synthetic Division (§ 572).

Transform into an equation whose roots are less by 5,

$$x^4 - 4x^3 - 7x^2 + 22x + 24 = 0.$$

$$\begin{array}{r} -4 - 7 + 22 + 24 \\ + 5 + 5 - 10 + 60 \\ + 1 - 2 + 12 + 84 \text{ (84 = first remainder).} \\ + 5 + 30 + 140 \\ + 6 + 28 + 152 \dots \text{ (152 = second remainder).} \\ + 5 + 55 \\ + 11 + 83 \dots \dots \dots \text{ (83 = third remainder).} \\ + 5 \\ + 16 \dots \dots \dots \text{ (16 = fourth remainder).} \end{array}$$

The required equation is, therefore,

$$y^4 + 16y^3 + 83y^2 + 152y + 84 = 0.$$

The roots of this equation are -1 , -2 , -6 , and -7 ; the roots of the original equation are 4 , 3 , -1 , and -2 .

EXERCISE 143.

Transform each equation below into another whose roots are less by the number placed in the parenthesis directly after the equation:

$$1. \quad x^3 - 11x^2 + 31x - 12 = 0. \quad (1.)$$

$$2. \quad x^4 - 6x^3 + 4x^2 + 18x - 5 = 0. \quad (2.)$$

$$3. \quad x^3 + 10x^2 + 13x - 24 = 0. \quad (-2.)$$

$$4. \quad x^3 - 9x^2 + 22x - 12 = 0. \quad (3.)$$

$$5. \quad x^4 + x^3 - 16x^2 - 4x + 48 = 0. \quad (4.)$$

$$6. \quad x^4 + 2x^3 - 25x^2 - 26x + 120 = 0. \quad (0.7.)$$

$$7. \quad x^4 + x^2 - 3x + 4 = 0. \quad (0.3.)$$

$$8. \quad x^5 + x^4 + 3x^3 - 2x - 16 = 0. \quad (0.5.)$$

$$9. \quad x^5 - 3x^4 - 2x^3 + 3x^2 - 7x + 12 = 0. \quad (-1.)$$

$$10. \quad x^6 - x^5 + 2x^4 - 3x^3 + 4x^2 - 5x + 6 = 0. \quad (0.2.)$$

601. To transform $f(x) = 0$ into an equation whose second term is wanting, substitute $y - \frac{A}{n}$ for x .

In (3), § 599, to make the second term vanish, nh must equal $-A$, and so h must equal $-\frac{A}{n}$. In that case, $x = y - \frac{A}{n}$. Hence, to make the second term vanish, substitute $y - \frac{A}{n}$ for x .

CHAPTER XXXIV.

HIGHER NUMERICAL EQUATIONS.

SITUATION OF THE ROOTS.

602. A **commensurable** root is rational, and is either integral or fractional.

An **incommensurable** root is a real root which is not commensurable.

In the solution of higher numerical equations, we first determine, by § 572, the commensurable roots. These are integral, when the equation is in the form $f(x) = 0$ (§ 585). In finding the first root, which we shall denote by a , we get the coefficients of the equation

$$\frac{f(x)}{x-a} = 0;$$

we then find a commensurable root, b , of that equation, and get the equation

$$\frac{f(x)}{(x-a)(x-b)} = 0.$$

Each division *depresses* the degree of the equation divided, and when all the commensurable roots have thus been divided out, if the resulting equation is of too high a degree to be solved easily, we proceed to find the incommensurable roots by Horner's Method. Imaginary roots may then be sought; the student, however, will not be required to search for these.

603. Find the commensurable roots of

$$x^6 + 3x^5 - 2x^4 - 15x^3 - 15x^2 + 8x + 20 = 0.$$

The factors of 20 are $\pm 1, \pm 2, \pm 4, \pm 5$, and ± 10 . We try 1 at first, as follows :

$$\begin{array}{r} +3-2-15-15+8+20 \\ +1+4+2-13-28-20 \\ \hline +4+2-13-28-20 \quad 0 \end{array}$$

Hence, 1 is a root; and the last line contains the coefficients (except the first) of the equation obtained by dividing the original equation by $x-1$. Next try -1 .

$$\begin{array}{r} +4+2-13-28-20 \\ -1-3+1+12+16 \\ \hline +3-1-12-16-4 \end{array}$$

Hence, -1 is not a root. A trial of 2 fails. We then try -2 .

$$\begin{array}{r} +4+2-13-28-20 \\ -2-4+4+18+20 \\ \hline +2-2-9-10 \quad 0 \end{array}$$

Hence, -2 is a root; and the last line contains the coefficients of the next depressed equation. A trial of 2 fails, but -2 succeeds, giving as the coefficients of the depressed equation

$$0, -2, -5.$$

Trials of the factors of -5 all fail. The roots so far found are 1, -2 , and -2 .

The equal roots might have been found by § 594, but the work would have been tedious. Three exact divisions have been made, depressing the degree of the original equation by three, and the last equation is, therefore,

$$x^3 + 0x^2 - 2x - 5 = 0,$$

whose roots are incommensurable.

EXERCISE 144.

Find the commensurable roots of each equation below. The number of these roots is given in the parenthesis.

$$1. \ x^4 - 4x^3 - 8x + 32 = 0. \quad (2.)$$

$$2. \ x^3 - 6x^2 + 10x - 8 = 0. \quad (1.)$$

$$3. \ x^4 + 2x^3 - 7x^2 - 8x + 12 = 0. \quad (4.)$$

$$4. \ x^3 + 3x^2 - 30x + 36 = 0. \quad (1.)$$

$$5. \ x^4 - 12x^3 + 32x^2 + 27x - 18 = 0. \quad (2.)$$

$$6. \ x^4 - 9x^3 + 17x^2 + 27x - 60 = 0. \quad (2.)$$

$$7. \ x^5 - 5x^4 + 3x^3 + 17x^2 - 28x + 12 = 0. \quad (5.)$$

$$8. \ x^4 - 10x^3 + 35x^2 - 50x + 24 = 0. \quad (4.)$$

$$9. \ x^5 - 8x^4 + 11x^3 + 29x^2 - 36x - 45 = 0. \quad (3.)$$

$$10. \ x^5 - x^4 - 6x^3 + 9x^2 + x - 4 = 0. \quad (1.)$$

604. After the commensurable roots are removed, the situation of the incommensurable roots is to be determined. Many methods of doing this have been discovered, the most noted of which is **Sturm's Theorem**; but the application is tedious, and it is generally best to proceed by trial.

605. *If two trial values of x , substituted in $f(x)$, give results with unlike signs, at least one root lies between the values. If the signs of the results are alike, no root, or some even number of roots, lies between the values.*

Let the trial-values of x be m and n , and the resulting values of $f(x)$ be M and N , which have unlike signs. If x be supposed to vary continuously from m to n , $f(x)$ will pass from M to N , and must pass through zero, because M and N have unlike signs.

The value of x which makes $f(x) = 0$ is a root; hence, at

least one root lies between m and n . $f(x)$ may pass through zero any odd number of times; but if the number of times is even, M and N will have like signs.

Let M and N have *like* signs. When x varies from m to n , $f(x)$ may not pass through zero at all. It *cannot* pass through it an *odd* number of times, for M and N would then have unlike signs. It *may* pass any even number of times; and therefore no root, or an even number of roots, lies between m and n .

These theorems are beautifully illustrated by the loci of equations. Thus, if any two abscissas are chosen (representing values of x), the signs of the corresponding ordinates, which represent values of $f(x)$, are unlike when the number of intersections of the curve with the axis of X is odd; the ordinates have like signs when the number of intersections is even.

606. *The value of $f(x)$, when $x=m$, is the remainder left after dividing $f(x)$ by $(x-m)$.*

Let Q be the quotient, and R the remainder which does not contain x . Then

$$f(x) = (x-m)Q + R.$$

Now, when m is substituted for x ,

$$f(x) \text{ or } f(m) = R.$$

Thus, when $x=5$, the value of

$$x^5 - 4x^4 + 7x^2 - 2x + 47$$

is computed by § 572 as follows:

$$\begin{array}{r} -4 + 0 + 7 - 2 + 47 \\ + 5 + 5 + 25 + 160 + 790 \\ \hline + 1 + 5 + 32 + 158 + 837 \end{array}$$

Therefore, $f(x) = 837$ when $x=5$, and this value is obtained more expeditiously than by direct substitution.

EXERCISE 145.

Compute the value of each expression below, the value of x being enclosed in the parenthesis following each:

1. $x^4 - 5x^3 + 26x^2 - 4x + 7$. (5.)
2. $x^3 - 4x^2 + 5x - 22$. (-7.)
3. $x^5 - 2x^4 + 3x^3 + x^2 - 28$. (2.)
4. $x^5 + 7x^3 - 2x^2 - 49$. (-3.)
5. $x^5 - 14x^3 + 473$. (6.)
6. $x^6 - 2x^5 + 3x^4 + 2x^3 + x^2 - 7x - 96$. (-2.)
7. $x^6 - x^5 - 2x^4 + x^3 - 6x + 14$. (3.)
8. $x^5 - 4x^4 + 2x^2 - 7x + 16$. (10.)
9. $x^7 - x^6 - 2x^5 - 3x^4 + 2x^3 + x^2 - x + 4$. (-2.)
10. $x^7 - 5x^5 + 6x^3 + 3x - 1$. (4.)

607. Let it be required to find the situation of the roots of

$$x^4 - 2x^3 - 11x^2 + 6x + 2 = 0.$$

They are real. Since there are two variations and two permanences of sign, there are two positive and two negative roots. Inspection of the signs and coefficients shows that these roots are small numbers.

The values of $f(x)$ may be computed by § 606, assuming values of x from -4 onward, until the situations of all the roots are discovered. These values are tabulated below:

If x is	$f(x)$ is	If x is	$f(x)$ is
-4	+186	+1	-4
-3	+20	+2	-30
-2	-22	+3	-52
-1	-12	+4	-22
0	+2	+5	+132

The changes of sign in the columns headed $f(x)$ show that the first root lies between -3 and -2 , the second between -1 and 0 , the third between 0 and $+1$, and the fourth between $+4$ and $+5$. When any root lies between 0 and ± 1 , before using Horner's Method, it is necessary to find the first significant figure.

Search for the root between 0 and $+1$:

$$\text{If } x=0.5, \quad f(x)=+2.1.$$

$$\text{" } x=0.6, \quad f(x)=+1.3.$$

$$\text{" } x=0.7, \quad f(x)=+0.4.$$

$$\text{" } x=0.8, \quad f(x)=-0.9.$$

The root therefore lies between 0.7 and 0.8 .

The root between 0 and -1 is next sought:

$$\text{If } x=-0.2, \quad f(x)=+0.4.$$

$$\text{" } x=-0.3, \quad f(x)=-0.7.$$

This root therefore lies between -0.2 and -0.3 .

The first significant figures of the other roots are evidently -2 and $+4$.

608. In the preceding example, the changes of sign of $f(x)$ led to the easy discovery of the situation of the roots. But two successive values of $f(x)$, having the same sign, may, by § 605, have two or some other even number of roots between them.

Take the equation

$$x^3 + 5x^2 - 7x + 2 = 0,$$

whose roots are all real. The law of signs gives two positive and one negative root. It is easy to see that the positive values of x do not exceed $+2$, and that the negative value is about -5 . We therefore tabulate values of $f(x)$.

If x is	$f(x)$ is	If x is	$f(x)$ is
+2	+16	-3	+41
+1	+1	-4	+46
0	+2	-5	+37
-1	+13	-6	+8
-2	+28	-7	-47

One root is $-6+$; the values of $f(x)$ indicate by their rapid diminution and close approach to zero in the first part of the table, that $f(x)$ becomes zero when x is between 0 and $+1$.

$$\text{If } x=0.4, \quad f(x)=+0.1.$$

$$\text{" } x=0.5, \quad f(x)=-0.1.$$

$$\text{" } x=0.6, \quad f(x)=-0.2.$$

$$\text{" } x=0.7, \quad f(x)=-0.1.$$

$$\text{" } x=0.8, \quad f(x)=+0.1.$$

The other roots are therefore $0.4+$ and $0.7+$.

609. The figure sought may also be found by a method which depends upon the principles of *differences*. Take the example in § 607. In the table below, the column headed d_1 contains the differences of the first order; that headed d_2 contains the differences of the second order; and so on. The differences which are of the same order as the degree of the equation, are constant, and furnish a test of the accuracy of the computation of the values of $f(x)$.

x	$f(x)$	d_1	d_2	d_3	d_4
-3	+20				
-2	-22	-42			
-1	-12	+10	+52		
0	+2	+14	+4	-48	
+1	-4	-6	-20	-24	+24
+2	-30	-26	-20	0	+24
+3	-52	-22	+4	+24	+24

In Example (3), § 507, $n = \frac{3}{4}$, and it is required to find the number which added to 6798.9 will give the answer. If the answer 8331.4 had been given and n required, the following equation might have been written :

$$8331.4 = 6798.9 + n(2029.7) + \frac{n(n-1)}{2}(123.1) + \dots,$$

or, in a general form,

$$A = a + na_1 + \frac{n(n-1)}{2}a_2 + \dots,$$

the solution of which would give an approximate value of n .

To find the first figure of the root which lies between 0 and +1 in the present example,

$$A = f(x) = 0; \quad a = +2; \quad a_1 = -6; \quad a_2 = -20.$$

The equation is

$$0 = 2 - 6n - 20 \frac{n(n-1)}{2} - \dots;$$

or, reducing,

$$10n^2 - 4n = 2;$$

whence $n = 0.7$, roughly; and, adding this to 0 (the value of x which stands opposite a), 0.7 is an approximate value of one of the roots.

The next figure of the root which lies between -3 and -2 may be obtained by means of the equation,

$$0 = 20 - 42n + 52 \frac{n(n-1)}{2};$$

or, reducing,

$$26n^2 - 68n = -20;$$

whence, $n = 0.3$, roughly; and, adding this to -3 (the value of x which stands opposite a), -2.7 is an approximate value of this root.

610. By applying principles already learned, and by the exercise of due discretion, the student will find little difficulty with the following examples. Skill is acquired only by practice.

EXERCISE 146.

Determine the first significant figure of each root in the following equations :

1. $x^3 - x^2 - 2x + 1 = 0.$
2. $x^3 - 5x - 3 = 0.$
3. $x^3 - 5x^2 + 7 = 0.$
4. $x^3 - 7x + 7 = 0.$
5. $x^3 + 2x^2 - 30x + 39 = 0.$
6. $x^3 - 6x^2 + 3x + 5 = 0.$
7. $x^3 + 9x^2 + 24x + 17 = 0.$
8. $x^3 - 15x^2 + 63x - 50 = 0.$
9. $x^4 - 8x^3 + 14x^2 + 4x - 8 = 0.$
10. $x^4 - 12x^2 + 12x - 3 = 0.$

611. In difficult cases, where the roots are widely different in value, Sturm's Theorem is sometimes useful; and it is given, without proof, simply for reference :

Divide out the equal roots of the equation, and let the resulting equation be $f(x) = 0$. Let $f_1(x)$ be the first derivative of $f(x)$. Perform the operations for finding the H. C. F. of $f_1(x)$ and $f(x)$, *changing, however, the sign of each remainder before using it as a divisor*. Denote these successive remainders thus changed by $f_2(x)$, $f_3(x)$,, and let r be the final remainder with its sign changed; this remainder does not contain x .

$f(x)$, $f_1(x)$, $f_2(x)$,, are called **Sturm's Functions**.

Let a and b be trial-values of x . Assuming that $x = a$, compute in order the value of Sturm's Functions, and note the number of variations of sign in these values; likewise, note the number of variations when $x = b$. The difference between the number of variations in the first case and the number in the second equals the number of real roots between a and b .

HORNER'S METHOD.

612. Having found one or more figures of the root, we employ for the remainder of the work **Horner's Method**, which was discovered in the early part of this century by W. G. Horner, of Bath, England. This process will be explained by the solution of an example.

Take the equation

$$x^3 - 6x^2 + 3x + 5 = 0. \quad (1)$$

Suppose that we have found, by one of the methods already given, that a root of this equation lies between 1 and 2. By the method of § 600, we transform the equation into one whose roots are less by 1.

$$\begin{array}{r} -6 \quad +3 \quad +5 \\ +1 \quad -5 \quad -2 \\ \hline -5 \quad -2 \quad +3 \\ +1 \quad -4 \\ \hline -4 \quad -6 \\ +1 \\ \hline -3 \end{array}$$

The transformed equation is, therefore,

$$y^3 - 3y^2 - 6y + 3 = 0; \quad (2)$$

or,

$$6y = 3 - 3y^2 + y^3.$$

Neglecting for the moment the terms $-3y^2 + y^3$, since (as will be seen presently) their numerical value is not large, we find $y = 0.5$. With this assumed value of y , computing the value of $-3y^2 + y^3$, and substituting, we obtain

$$6y = 2.375;$$

whence $y = 0.4$, approximately. We next transform (2) into an equation whose roots are 0.4 less.

-3	-6	+3
<u>+0.4</u>	<u>-1.04</u>	<u>-2.816</u>
-2.6	-7.04	+0.184
<u>+0.4</u>	<u>-0.88</u>	
-2.2	-7.92	
<u>+0.4</u>		
-1.8		

The second transformed equation, whose roots are 0.4 less than those of (2), and 1.4 less than those of (1), is, therefore,

$$z^3 - 1.8z^2 - 7.92z + 0.184 = 0. \quad (3)$$

Neglecting the first two terms, because z is a small decimal, and $z^3 - 1.8z^2$ is still smaller, we find $z = 0.02 +$.

Equation (3) is now to be transformed into an equation whose roots are 0.02 less.

-1.8	-7.92	+0.184
<u>+0.02</u>	<u>-0.0356</u>	<u>-0.159112</u>
-1.78	-7.9556	+0.024888
<u>+0.02</u>	<u>-0.0352</u>	
-1.76	-7.9908	
<u>+0.02</u>		
-1.74		

The third transformed equation, whose roots are 1.42 less than those of (1), is, therefore,

$$v^3 - 1.74v^2 - 7.9908v + 0.024888 = 0; \quad (1)$$

whence $v = 0.003 +$. As the root of (4) is 1.42 less than the root of (1), the root of (1) must be $1.42 + 0.003 +$, or $1.423 +$. The preceding process can be continued until the root of (1) is found to any required degree of accuracy.

Since the values of y, z, v , etc., are to be added to the first rough value of x , they are all positive; and hence the

coefficient of the first power of the unknown quantity in each transformed equation is unlike in sign to the absolute term. If, in (4), the signs of the last two terms were alike, the value of v would be $-0.003+$. This would show that the value assumed for z was too great, and we should diminish the value of z and make the last transformation again. In beginning an example, one is very likely to assume too large a value for the next figure of the root; in solving (2), for instance, the first solution gave $y=0.5$, and, had that value been tried, it would have proved to be too great.

613. It is not necessary to write out the transformed equations in full, as in the last section. But when the coefficients have been computed, the next figure of the root may be found by dividing the last coefficient (the absolute term), with its sign changed, by the coefficient which precedes it. Thus, instead of writing out (4) § 612, we might have obtained the value of v by simply dividing $-0.0248+$ by $-7.99+$. Indeed, the following will be found to be a good practical rule:

Whenever a true significant decimal figure of the root has been found by trial, and the figure found by dividing the last coefficient by the coefficient which precedes it proves to be the same, the subsequent figures of the root may also be successively found by dividing the last coefficients of the succeeding transformed equations by the coefficients which precede them.

If n decimal places are required in the root, the numbers in the last column of the following scheme need not be carried further than the n th decimal place. In the other columns, a sufficient number of decimals must be employed to insure the accuracy of the n th place of the last column.

- 6	+ 3	+ 5	1.423 +
<u>1</u>	<u>- 5</u>	<u>- 2</u>	
- 5	<u>- 2</u>	<u>3</u>	
<u>1</u>	<u>- 4</u>		- 2.816
- 4	<u>- 6</u>		<u>0.184</u>
<u>1</u>	<u>- 1.04</u>		<u>- 0.159112</u>
- 3	<u>- 7.04</u>		<u>0.024888</u>
<u>0.4</u>	<u>- 0.88</u>		<u>- 0.023988</u>
- 2.6	<u>- 7.92</u>		<u>0.000900</u>
<u>0.4</u>	<u>- 0.0356</u>		
- 2.2	<u>- 7.9556</u>		
<u>0.4</u>	<u>- 0.0352</u>		
- 1.8	<u>- 7.9908</u>		
<u>0.02</u>	<u>- 0.0052</u>		
- 1.78	<u>- 7.9960</u>		
<u>0.02</u>	<u>- 0.0052</u>		
- 1.76	<u>- 8.0012</u>		
<u>0.02</u>			
- 1.74			
<u>0.003</u>			
- 1.737			
<u>0.003</u>			
- 1.734			
<u>0.003</u>			
- 1.731			

The coefficients of each transformed equation are readily discerned by observing the lines which separate the different parts of the work.

The next three figures of the root can now be found by dividing -0.000900 by -8.0012 , giving $0.000112+$. The reason for this is shown by consideration of the last transformed equation,

$$w^3 - 1.731w^2 - 8.0012w + 0.000900 = 0.$$

As the value of w is about 0.0001 , the value of $w^3 - 1.731w^2$ is only about -0.00000002 ; therefore we may assume, without sensible error, that

$$8.0012w = 0.000900;$$

whence

$$w = 0.000112,$$

as above.

614. One of the roots of the equation of § 612 is $-0.6 +$; and to avoid, as far as possible, the inconvenience of computing with negative numbers, it is customary to transform the equation into one whose root is $+0.6 +$. By § 596, the transformed equation is

$$x^3 + 6x^2 + 3x - 5 = 0,$$

a root of which is $0.66966 +$; hence, a root of the original equation is $-0.66966 +$.

615. Sometimes the coefficient of the first power of the unknown number in one of the transformed equations is zero. To find the next figure of the root in this case, divide the absolute term by the coefficient of the second power of the unknown number, and extract the square root of the quotient.

For instance,

$$y^4 - 7.6y^3 + 3.64y^2 + 0y - 0.0007829 = 0.$$

whence, $3.64y^2 = 0.0007829$, approximately,
and $y = 0.01 +$.

616. In rare cases two of the roots are so nearly equal that Horner's Method, carried out as above, will not find them both. Take, for example,

$$x^3 + 11x^2 - 102x + 181 = 0,$$

in which we have found that there are two roots between 3 and 4. Horner's Method gives for the first transformed equation

$$y^3 + 20y^2 - 9y + 1 = 0.$$

Since the coefficient of y^2 is large, it is best to neglect y^3 only; and the solution gives $y = 0.2$.

The next transformed equation is

$$z^3 + 20.6z^2 - 0.88z + 0.008 = 0,$$

and $-0.008 \div -0.88 = 0.009 +$.

Perceiving that $20.6z^2$, when transposed, will increase the right-hand member numerically, we conclude to assume 0.01 as the next portion of the root. Proceeding in the usual manner, we find one value of x to be $3.21312 +$. The first two decimals of the other value of x are found by trial.

$$\text{If } x = 3.21, \quad f(x) = +0.001261.$$

$$\text{" } x = 3.22, \quad f(x) = -0.001352.$$

$$\text{" } x = 3.23, \quad f(x) = +0.000167.$$

The change of sign, and the numerical values of $f(x)$, show that the second value of x is nearly 3.229.

The application of the usual process gives $x = 3.22952 +$.

617. When the root sought is a large number, in finding the successive figures of its *integral* portion we do not divide the absolute term by the coefficient of the first power of the unknown number, because the neglect of the higher powers, which are in this case large numbers, would lead to serious error.

Let it be required to find one root of

$$x^4 - 3x^3 + 11x - 4842624131 = 0. \quad (1)$$

By trial, we find that a root lies between 200 and 300. Diminishing the roots of (1) by 200, we have

$$y^4 + 800y^3 + 239997y^2 + 31998811y - 3242741931 = 0. \quad (2)$$

$$\text{If } y = 60, \quad f(y) = -273064071.$$

$$\text{" } y = 70, \quad f(y) = +471570139.$$

The signs of $f(y)$ show that a root lies between 60 and 70. Diminishing the roots of (2) by 60, we obtain

$$z^4 + 1040z^3 + 405597z^2 + 70302451z - 273064071 = 0. \quad (3)$$

The root of this is found by trial to lie between 3 and 4. Diminishing the roots by 3, we may find the remaining figures of the root of the original equation, which are decimal, by the usual process.

The scheme of the work is as follows :

0	— 3	+ 11	— 4842624131
<u>200</u>	<u>40000</u>	<u>7999400</u>	<u>1599882200</u>
200	39997	7999411	— 3242741931
<u>200</u>	<u>80000</u>	<u>23999400</u>	<u>2969677860</u>
400	119997	31998811	— 273064071
<u>200</u>	<u>120000</u>	<u>17495820</u>	<u>214585887</u>
600	239997	49494631	— 58478184
<u>200</u>	<u>51600</u>	<u>20807820</u>	<u>58477522.9</u>
800	291597	70302451	— 661.1
<u>60</u>	<u>55200</u>	<u>1226178</u>	
860	346797	71528629	
<u>60</u>	<u>58800</u>	<u>1235592</u>	
920	405597	72764221	
<u>60</u>	<u>3129</u>	<u>332682.6</u>	
980	408726	73096903.6	
<u>60</u>	<u>3138</u>	<u>333356.9</u>	
1040	411864	73430260.5	
<u>3</u>	<u>3147</u>		
1043	415011		
<u>3</u>	<u>842.2</u>		
1046	415853.2		
<u>3</u>	<u>842.9</u>		
1049	416696.1		
<u>3</u>	<u>843.5</u>		
1052	417539.6		
<u>0.8</u>			
1052.8			
<u>0.8</u>			
1053.6			
<u>0.8</u>			
1054.4			
<u>0.8</u>			
1055.2			

$\frac{661.1}{73430260.5} = 0.000009+.$
$x = 263.800009+.$

618. Any root of a number can be extracted by Horner's Method. Find the fourth root of 473.

Here $x^4 = 473$,
 or $x^4 + 0x^3 + 0x^2 + 0x - 473 = 0$,
 and $x = 4.66353+.$

If the number be a perfect power, the root will be obtained exactly.

619. From the preceding sections are derived the following general directions for solving a higher numerical equation whose roots are real :

I. Remove commensurable roots by § 603.

II. By inspection and Descartes' Rule find roughly the situations of the incommensurable roots.

III. Find one or more figures of each incommensurable root by §§ 607-609.

IV. Apply Horner's Method as set forth in §§ 612-618.

620. Compute one root of each of the following equations, carrying each result to six decimal places. The last two or three places may be found as in the last part of § 613. The root to be sought lies between the two numbers placed in the parenthesis following each equation, except in examples 8-12.

EXERCISE 147.

$$1. x^3 + 10x^2 + 6x - 120 = 0. \quad (2, 3.)$$

$$2. x^3 + x^2 + x - 100 = 0. \quad (4, 5.)$$

$$3. x^4 - 2x^3 + 21x - 23 = x. \quad (1, 2.)$$

$$4. x^4 - 5x^3 + 3x^2 + 35x - 70 = 0. \quad (2, 3.)$$

$$5. x^4 - 12x^2 + 12x - 3 = 0. \quad (-3, -4.)$$

$$6. x^5 + 2x^4 + 3x^3 + 4x^2 + 5x - 54321 = 0. \quad (8, 9.)$$

$$7. x^4 - 59x^2 + 840 = 0. \quad (4, 5.)$$

$$8. x^3 - 35499 = 0. \quad 11. x^5 - 147008443 = 0.$$

$$9. x^3 - 242970624 = 0. \quad 12. x^2 - 551791 = 0.$$

$$10. x^4 - 707281 = 0. \quad 13. x^2 - 17x + 70.3 = 0. \quad (7, 8.)$$

$$14. x^3 + 9x^2 + 24x + 17 = 0. \quad (-4, -5.)$$

$$15. x^4 - 8x^3 + 14x^2 + 4x - 8 = 0. \quad (0, -1.)$$

621. Many methods have been found of abbreviating the computation, when it is desired to get the root to a large number of decimal places. We give one simple way which can sometimes be profitably used after three or four decimal places have been computed.

Consider the example in § 613. If that computation had not been abbreviated, the last transformed equation would have been

$$w^3 - 1.731w^2 - 8.001213w + 0.000899967 = 0. \quad (1)$$

We have found that w is between 0.00011 and 0.00012. Computing the value of $w^3 - 1.731w^2$ with these values of w , and substituting the results in (1), we have

$$8.001213w = 0.000899946 \pm, \quad (2)$$

$$\text{and} \quad 8.001213w = 0.000899942 \pm. \quad (3)$$

$$\text{From (2),} \quad w = 0.0001124762 \pm. \quad (4)$$

$$\text{From (3),} \quad w = 0.0001124757 \pm. \quad (5)$$

Since w is about one-fourth the way from 0.00011 to 0.00012, take a value one-fourth the way from the one in (4) to that in (5), and

$$w = 0.0001124761 \pm.$$

If we wish it more accurately still, $w^3 - 1.731w^2$ can be recomputed with the value of w just found; and substitution in (1) will give a value of w extending a few decimal places further; and so on.

622. When all but two roots of any equation have been determined, § 576 enables us to get both these roots at once.

Take the equation

$$x^3 - 6x^2 + 3x + 5 = 0,$$

one root of which is 1.423112. Let this root be represented by a , and the other two roots by b and c .

By § 576, $a + b + c = 6$, and $abc = -5$.

$$\therefore b + c = 4.576888,$$

$$bc = -3.513427,$$

$$b^2 + 2bc + c^2 = 20.94790,$$

$$4bc = -14.05371,$$

$$b^2 - 2bc + c^2 = 35.00161,$$

$$b - c = 5.916216,$$

$$2b = 10.493104,$$

$$2c = -1.339328,$$

$$b = 5.246552,$$

$$\text{and } c = -0.669664.$$

The above solution involves little labor when the computation is made with the aid of logarithms.

Two roots of

$$x^4 - 16x^3 + 79x^2 - 140x + 58 = 0$$

are 0.58579 and 8.64575; find the others.

SOLUTIONS OF SPECIAL FORMS.

RECURRING EQUATIONS.

623. A **Recurring Equation**, or **Reciprocal Equation**, is one in which the coefficients of terms equally distant from the extremes are numerically equal; the signs of the corresponding terms are either all alike or all unlike.

Below are examples of recurring equations:

$$x^5 - 3x^4 + 5x^3 - 5x^2 + 3x - 1 = 0,$$

$$x^4 - 7x^3 + 15x^2 - 7x + 1 = 0,$$

$$x^6 - 3x^5 + 5x^4 - 5x^3 + 3x^2 - 1 = 0.$$

The last equation shows that when the degree of the equation is even, and the signs of the corresponding terms unlike, the middle term is wanting. By definition, the middle term of such an equation is both positive and negative, and therefore must be zero.

624. *If a be a root of a recurring equation, $\frac{1}{a}$ is also a root.* For, in order to obtain an equation whose roots are reciprocals of those of the given equation, we simply write the coefficients in reverse order (§ 598); this, however, does not alter a recurring equation (except to change all its signs when in one form), and so, if a is a root of it, $\frac{1}{a}$ is also a root.

Recurring equations are called **Reciprocal Equations**, because of this property.

625. *A recurring equation of an odd degree has for one root -1 or $+1$, according as the signs of the corresponding terms are alike or unlike.*

$$x^n + Ax^{n-1} + Bx^{n-2} + \dots \pm Bx^2 \pm Ax \pm 1 = 0 \quad (1)$$

is a typical recurring equation, and may be written in the form

$$(x^n \pm 1) + A(x^{n-1} \pm x) + B(x^{n-2} \pm x^2) + \dots = 0. \quad (2)$$

When the signs of the corresponding terms are alike, we consider the upper signs in (2), since the coefficient x^n is positive. As n is an odd number, the substitution of -1 for x will satisfy the equation, because the expression in each parenthesis becomes zero. Likewise, when the signs of the corresponding terms are unlike, we must consider the lower signs in (2). The value $+1$ substituted for x will then satisfy the equation.

626. *A recurring equation of an even degree has $+1$ and -1 as roots, when the corresponding terms have unlike signs.*

Equation (2) of § 625 may be written, when the corresponding terms have unlike signs, as follows :

$$(x^n - 1) + Ax(x^{n-2} - 1) + Bx^2(x^{n-4} - 1) + \dots = 0.$$

In this case n is an even number, and every exponent of x is even. Each parenthesis is therefore divisible by x^2-1 , and the whole equation is divisible by $(x+1)(x-1)$; and consequently $+1$ and -1 are roots.

627. *A recurring equation of an even degree is reducible to an equation of half its degree, when the corresponding terms have like signs.*

The first and last terms are positive, but the intermediate terms need not be. For convenience, we shall represent them all as positive, and write the equation

$$x^{2n} + Ax^{2n-1} + Bx^{2n-2} + \dots + Bx^2 + Ax + 1 = 0. \quad (1)$$

Dividing (1) by x^n , and rearranging it, we have

$$\left(x^n + \frac{1}{x^n}\right) + A\left(x^{n-1} + \frac{1}{x^{n-1}}\right) + B\left(x^{n-2} + \frac{1}{x^{n-2}}\right) + \dots = 0. \quad (2)$$

The middle term of (1), since it is of the form Ex^n , will appear as a known number (E) in (2).

$$\text{Let} \quad \left(x + \frac{1}{x}\right) = y.$$

$$\text{Then} \quad \left(x^2 + \frac{1}{x^2}\right) = \left(x + \frac{1}{x}\right)^2 - 2 = y^2 - 2;$$

$$\left(x^3 + \frac{1}{x^3}\right) = \left(x + \frac{1}{x}\right)^3 - 3\left(x + \frac{1}{x}\right) = y^3 - 3y;$$

$$\left(x^4 + \frac{1}{x^4}\right) = \left(x^2 + \frac{1}{x^2}\right)^2 - 2 = y^4 - 4y^2 + 2; \quad \text{and so on.}$$

In general,

$$\left(x^n + \frac{1}{x^n}\right) = y^n - ny^{n-2} \dots$$

Hence, (2) may be expressed in terms of y , involving no power higher than the n th, and being, therefore, of half the degree of (1).

628. *Any recurring equation is reducible to one of half its degree.*

For the equations of § 625, being divided respectively by $x+1$ and $x-1$, are reduced to the form of § 627; the equation of § 626, being divided by x^2-1 , is reduced to the form of § 627. These reduced equations can now be treated by the method of § 627.

$$\text{Solve } x^6 - 11x^4 + 17x^3 + 17x^2 - 11x + 1 = 0.$$

One root is -1 , and dividing the equation by $x+1$, we have

$$x^4 - 12x^3 + 29x^2 - 12x + 1 = 0.$$

Divide by x^2 , rearrange the terms, and add 2 to both sides.

$$\text{Then } \left(x + \frac{1}{x}\right)^2 - 12\left(x + \frac{1}{x}\right) + 29 = 2;$$

$$\text{whence } \left(x + \frac{1}{x}\right) = 9 \text{ or } 3,$$

$$\text{and } x = \frac{1}{2}(9 \pm \sqrt{77}) \text{ or } \frac{1}{2}(3 \pm \sqrt{5}).$$

$\frac{1}{2}(9 + \sqrt{77})$ is the reciprocal of $\frac{1}{2}(9 - \sqrt{77})$, and $\frac{1}{2}(3 + \sqrt{5})$ is the reciprocal of $\frac{1}{2}(3 - \sqrt{5})$, the product in each case being unity.

EXERCISE 148.

Solve:

1. $x^4 + 7x^3 - 7x - 1 = 0.$
2. $x^4 + x^3 + x^2 + x + 1 = 0.$
3. $x^6 - 3x^5 + 5x^4 - 5x^2 + 3x - 1 = 0.$
4. $x^4 - 5x^3 + 6x^2 - 5x + 1 = 0.$
5. $2x^4 - 5x^3 + 6x^2 - 5x + 2 = 0.$
6. $x^5 - 4x^4 + x^3 + x^2 - 4x + 1 = 0.$
7. $x^4 - 10x^3 + 26x^2 - 10x + 1 = 0.$
8. $x^3 + mx^2 + mx + 1 = 0.$
9. $x^5 + 1 = 0.$
10. $3x^5 - 2x^4 + 5x^3 - 5x^2 + 2x - 3 = 0.$

EXPONENTIAL EQUATIONS.

629. An **Exponential Equation** is one in which an unknown number appears as an exponent.

I. Solve $a^x = m$. Taking the logarithms of both members, we have $x \log a = \log m$; whence $x = \frac{\log m}{\log a}$.

II. Solve $x^x = m$. As before, $x \log x = \log m$. Find by trial, from a logarithm table, a value of x which satisfies the equation. Thus, in the equation $x^x = 7$:

By trial, $2.3 \log 2.3 = 0.83197$;
 $2.32 \log 2.32 = 0.84793$;
 $2.316 \log 2.316 = 0.84474$. Hence, $x = 2.316$.

EXERCISE 149.

Solve:

- | | | |
|-------------------|----------------------|-----------------------|
| 1. $11^x = 346$. | 5. $4^x = 3.74$. | 8. $14.74^x = 8.64$. |
| 2. $3^x = 10$. | 6. $146^x = 12984$. | 9. $x^x = 2.767$. |
| 3. $10^x = 745$. | 7. $0.2^x = 0.4$. | 10. $x^x = 23.10$. |
| 4. $7^x = 324$. | | |

In Compound Interest, if P denotes the principal, A the amount, r the rate, t the integral number of years (or other periods of time), we have the formula,

$$A = P(1 + r)^t;$$

whence

$$(1 + r)^t = \frac{A}{P},$$

and

$$t = \frac{\log A - \log P}{\log (1 + r)}.$$

Find t in the following cases:

- | | | |
|---------------------|------------------|------------------------|
| 11. $P = 750$, | $A = 1797.42$, | $r = 6\%$. |
| 12. $P = 780$, | $A = 1559.22$, | $r = 8\%$. |
| 13. $P = 5630.75$, | $A = 21789.22$, | $r = 7\%$. |
| 14. $P = 300$, | $A = 515.46$, | $r = 7\%$. |
| 15. $P = 84.65$, | $A = 289.47$, | $r = 7\frac{1}{2}\%$. |

CARDAN'S METHOD FOR CUBIC EQUATIONS.

630. The general form of a cubic equation is

$$x^3 + mx^2 + nx + c = 0. \quad (1)$$

Let $x = y - \frac{m}{3}.$ (2)

By § 601, equation (1) reduces to the form

$$y^3 + py + q = 0. \quad (3)$$

Let $y = z - \frac{p}{3z}.$ (4)

Substitute this value of y in (3) and reduce:

$$z^6 + qz^3 = \frac{p^3}{27},$$

whence $z^3 = -\frac{1}{3}q \pm \sqrt{\frac{p^3}{27} + \frac{q^2}{4}},$

and $z = \sqrt[3]{-\frac{1}{3}q \pm \sqrt{\frac{p^3}{27} + \frac{q^2}{4}}}. \quad (5)$

From (2) and (4), $x = z - \frac{p}{3z} - \frac{m}{3}. \quad (6)$

When z has been determined from (5), x can be found from (6). In solving these equations, it is best to get from (5) the value of z most easily obtained. Then find the value of x from (6), and designate this value by r ; then divide (1) by $x - r$, and solve the resulting quadratic. Or the method of § 622 may be employed. If the original equation does not contain the square of the unknown number, it will be in the form of (3), and the value of y can be found from (5) and (4).

Solve $x^3 - 6x^2 - 12x + 112 = 0.$

By (2), $x = y - \frac{-6}{3} = y + 2.$

Substitution gives $y^3 - 24y + 72 = 0.$

Here $p = -24,$ and $q = 72.$

$$\begin{aligned}\therefore \text{from (5),} \quad z &= \sqrt[3]{-36 \pm \sqrt{-\frac{13824}{27} + \frac{5184}{4}}} \\ &= \sqrt[3]{-36 \pm 28} = -2.\end{aligned}$$

$$\text{From (6),} \quad x = -5.$$

Divide the original equation by $x + 4$; the resulting equation is

$$x^2 - 10x + 28 = 0;$$

$$\text{whence} \quad x = 5 \pm \sqrt{-3}.$$

631. If p is negative, and p and q have such values that $\frac{p^3}{27} + \frac{q^2}{4}$ is negative, equation (5) shows that Cardan's Method is unsatisfactory, since z is the cube root of an expression which is partly imaginary, and we have no general rule for extracting such a root. This case, which is called the *irreducible case*, can be solved by Trigonometry, and may be illustrated by an example.

$$\text{Solve} \quad y^3 - 15y - 4 = 0.$$

Substitution in (5) gives

$$z = \sqrt[3]{2 \pm \sqrt{\frac{-3375}{27} + 4}} = \sqrt[3]{2 \pm 11\sqrt{-1}}.$$

It is possible to determine, by trial, that

$$(2 \pm \sqrt{-1})^3 = 2 \pm 11\sqrt{-1};$$

$$\therefore z = 2 \pm \sqrt{-1},$$

$$\text{and, from (4),} \quad y = 4.$$

Divide the given equation by $y - 4$, and solve the resulting quadratic; then

$$y = -2 \pm \sqrt{3}.$$

Solve : EXERCISE 150.

$$1. \quad x^3 + 12x^2 + 45x + 50 = 0.$$

$$5. \quad y^3 + 48y + 504 = 0.$$

$$2. \quad x^3 - 21x^2 + 159x - 490 = 0.$$

$$6. \quad y^3 - 21y - 344 = 0.$$

$$3. \quad x^3 - 6x^2 + 13x - 10 = 0.$$

$$7. \quad y^3 - 3y + 2 = 0.$$

$$4. \quad x^3 + 3x^2 + 9x - 13 = 0.$$

$$8. \quad y^3 - 60y + 671 = 0.$$

TRIGONOMETRIC SOLUTION OF CUBIC EQUATIONS.

632. Trigonometric formulas are used in solving quadratic and cubic equations; and we subjoin those for cubics, so that the student may use them when he has acquired a knowledge of Trigonometry.

In the formulas, p is essentially positive, but q may be either positive or negative. We distinguish three cases:

- I. $x^3 + px + q = 0$;
 II. $x^3 - px + q = 0$, and $\frac{p^3}{27} < \frac{q^2}{4}$;
 III. $x^3 - px + q = 0$, and $\frac{p^3}{27} > \frac{q^2}{4}$.

The formulas for Case I. are as follows:

$$\tan \theta = \frac{2p}{3q} \sqrt{\frac{p}{3}}. \quad \tan \frac{1}{3} \phi = \sqrt[3]{\tan \frac{1}{3} \theta}.$$

$$x_1 = -2 \sqrt{\frac{p}{3}} \cot \phi.$$

$$x_2 = \sqrt{\frac{p}{3}} \cot \phi + \csc \phi \sqrt{p} \sqrt{-1}.$$

$$x_3 = \sqrt{\frac{p}{3}} \cot \phi - \csc \phi \sqrt{p} \sqrt{-1}.$$

The formulas for Case II. are as follows:

$$\sin \theta = \frac{2p}{3q} \sqrt{\frac{p}{3}}. \quad \tan \frac{1}{3} \phi = \sqrt[3]{\tan \frac{1}{3} \theta}.$$

$$x_1 = -2 \sqrt{\frac{p}{3}} \csc \phi.$$

$$x_2 = \sqrt{\frac{p}{3}} \csc \phi + \cot \phi \sqrt{p} \sqrt{-1}.$$

$$x_3 = \sqrt{\frac{p}{3}} \csc \phi - \cot \phi \sqrt{p} \sqrt{-1}.$$

The formulas for Case III. are as follows :

$$\sin \theta = \frac{3q}{2p} \sqrt{\frac{3}{p}}.$$

$$x_1 = 2 \sqrt{\frac{p}{3}} \sin \frac{1}{3} \theta.$$

$$x_2 = 2 \sqrt{\frac{p}{3}} \sin (60^\circ - \frac{1}{3} \theta).$$

$$x_3 = -2 \sqrt{\frac{p}{3}} \sin (60^\circ + \frac{1}{3} \theta).$$

If $\sin \theta$ or $\tan \theta$ is positive, θ is in the first quadrant ; if it is negative, θ is in the fourth quadrant.

In the preceding formulas, x_1 , x_2 , x_3 denote the three roots in each case.

633. Take the difficult equation

$$x^3 - \frac{403}{44}x + \frac{46}{147} = 0,$$

which comes under Case III.

log 403 = 2.60531	$\theta = 68^\circ 32' 0''$
log 441 = 2.64444	$\frac{1}{3} \theta = 22^\circ 50' 40''$
log 46 = 1.66276	log sin $\frac{1}{3} \theta = 9.58909 - 10$
log 147 = 2.16732	log 2 $\sqrt{\frac{p}{3}} = 0.04291$
log 2 = 0.30103	log sin $(60^\circ - \frac{1}{3} \theta) = 9.78102 - 10$
log $p = 9.96087 - 10$	log sin $(60^\circ + \frac{1}{3} \theta) = 9.99660 - 10$
log 3 = 0.47712	log $x_1 = 9.63200 - 10$
log $q = 9.49544 - 10$	log $x_2 = 9.82393 - 10$
log $3q = 9.97256 - 10$	log $(-x_3) = 0.03951$
colog $2p = 9.73810 - 10$	$x_1 = 0.42855$
log $\sqrt{\frac{3}{p}} = 0.25812$	$x_2 = 0.06670$
log sin $\theta = 9.96878 - 10$	$x_3 = -1.09525$

This is an unfavorable example, because $\sin \theta$ is large, and θ is therefore inaccurately determined; yet the values of the roots are near the true values, which are 0.42857, 0.66667, and -1.09524 . If $\log \sqrt{\frac{3}{p}}$, which was 0.258125, had been assumed to be 0.25813, the answers would have been 0.42857, 0.66668, and -1.09525 .

EXERCISE 151.

Solve :

1. $x^3 + 3x - 5 = 0$.

3. $x^3 - 7x + 11 = 0$.

2. $x^3 + 7x + 3 = 0$.

4. $x^3 - 4x - 5 = 0$.

5. $x^3 - 5x + 4 = 0$.



